Université Paris Diderot - Paris VII
UFR de Mathématiques

Année 2009

Thèse

pour obtenir le titre de

Docteur de l'Université Paris 7

Spécialité : Mathématiques Appliquées

présentée par

Idris KHARROUBI

EDS Rétrogrades et Contrôle Stochastique Séquentiel en Temps Continu en Finance

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Directeur de thèse

Pr. Huyên PHAM

Soutenue publiquement le 1er Décembre 2009, devant le jury composé de :

Pr. BOUCHARD Bruno, Université Paris 9 - Dauphine
Pr. EL KAROUI Nicole, Université Paris 6 - Pierre et Marie Curie
Pr. HAMADENE Saïd, Université du Maine
Pr. JEANBLANC Monique, Université d'Evry Val d'Essonne
Pr. LAMBERTON Damien, Université Paris-Est Marne-La-Vallée
Pr. PHAM Huyên, Université Paris 7 - Paris Diderot

au vu des rapports de :

Pr. HU Ying, Université Rennes 1
Pr. SONER Mete Halil, ETH Zürich

A ma famille

Remerciements

Mes premiers remerciements vont naturellement à mon directeur de thèse, Huyên Pham, sans qui cette thèse n'aurait jamais pu voir le jour. Je le remercie de m'avoir initié à la recherche dans le domaine des mathématiques financières et du contrôle stochastique. Je le remercie également pour ses nombreux conseils, son soutien constant, ainsi que pour la rigueur mathématique qu'il m'a apportée. Je lui suis particulièrement reconnaissant pour sa disponibilité tout au long de ces dernières années.

Je suis très reconnaissant envers Ying Hu et Halil Mete Soner d'avoir accepté de rapporter cette thèse. Je remercie également Bruno Bouchard, Nicole El Karoui, Saïd Hamadène, Monique Jeanblanc et Damien Lamberton d'avoir accepté de participer au jury.

Je suis également reconnaissant envers Jean-François Chassagneux, Romuald Elie, Jin Ma et Jianfeng Zhang pour leurs collaborations scientifiques.

J'exprime ma reconnaissance envers Michèle Wasse pour toutes ses aides administratives. Sans elle, le début de l'année scolaire de chaque étudiant ressemblerait certainement à un parcours du combattant.

Je remercie les thésards et anciens thésards que j'ai pu côtoyer au bâtiment Chevaleret, au CREST, à l'ENSAE ainsi qu'aux séminaires Bachelier et Bachelier des doctorants.

J'exprime toute ma reconnaissance à ma famille, un microcosme bien agréable, et tout particulièrement à mes parents.

Enfin, je termine par remercier celle qui, malgré la distance, m'a toujours soutenu et encouragé : mon épouse Salwa.

TABLE DES MATIÈRES

INTRODUCTION GÉNÉRALE

Cette thèse se compose de trois parties qui portent sur l'étude de certains problèmes d'optimisation stochastique et de leurs applications en mathématiques financières.

La première partie étudie la représentation par des Equations Différentielles Stochastiques Rétrogrades (EDSR) de solutions de problèmes d'optimisation stochastique séquentielle en temps continu. Il s'agit de problème d'optimisation où les quantités évoluent en temps continu mais où le contrôle est discret en temps : il consiste en une suite d'interventions. Nous nous intéressons à deux de ces problèmes : le contrôle impulsionnel et le *switching* optimal.

Nous adoptons une nouvelle approche consistant à voir les inéquations (quasi-) variationnelles dont sont solutions les fonctions valeurs de tels problèmes d'optimisation comme des problèmes contraints avec valeurs terminales données. Cette interprétation offre un cadre naturel à l'étude des EDSRs contraintes.

Cette nouvelle classe d'EDSRs est mise ensuite en lien avec les EDSRs à réflexions obliques, liées au problème de *switching* optimal, récemment introduites par Hu et Tang [44] puis généralisées par Hamadène et Zhang [41].

La seconde partie étudie l'approximation des EDSRs associées aux systèmes d'inéquations variationnelles.

Nous nous intéressons d'abord à l'approximation des EDSRs contraintes à sauts. La méthode de pénalisation utilisée pour la construction des solutions nous permet de mettre en place une procédure numérique pour la résolution de systèmes d'inéquations variationnelles fortement couplées.

Nous étudions ensuite les EDSRs à réflexions obliques et discrètes. Nous montrons qu'elles constituent une approximation des EDSRs à réflexions obliques de Hu et Tang [44]. Ces EDSRs à réflexions obliques discrètes permettent de mettre en place un schémas naturel de discrétisation pour lequel on obtient une vitesse de convergence.

La troisième partie traite d'un modèle de liquidation de portefeuille sous contrainte de coût et de risque d'exécution. Nous considérons un marché financier sur lequel un agent doit liquider une position en un actif risqué *i.e* ne plus posséder de part en cet actif risqué au bout d'une échéance fixée. L'intervention de cet agent influe sur le prix de marché de cet actif et conduit à un coût d'exécution lié à la taille du volume échangé d'une part, et à la fréquence d'intervention de l'agent sur le marché d'autre part. Ces phénomènes étant dus en pratique à l'épuisement du carnet d'ordre. Nous caractérisons la fonction valeur de notre problème comme solution minimale d'une inéquation quasi-variationnelle au sens de la viscosité contrainte.

0.1 Première Partie : Représentations probabilistes des solutions de problèmes d'optimisation stochastique séquentielle en temps continu

Cette partie traite de la représentation par EDSRs, de fonctions valeurs de problèmes d'optimisation stochastique séquentielle en temps continu : le contrôle impulsionnel et le *switching* optimal.

Ces problèmes d'optimisation séquentielle en temps continu, ont connu depuis quelques années un regain d'intérêt, notamment dans le monde financier, puisqu'ils permettent de mettre en place des modélisations relativement proche de la réalité : risque de liquidité sur les marchés financiers [42], [53] , gestion d'énergie [18], etc.

Nous considérons donc une nouvelle classe d'EDSR de la forme

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s . dW_s \\ &\quad - \int_t^T \int_E U_s(e) \mu(de, ds) + K_T - K_t, \end{aligned} \tag{0.1.1}$$

où W est un mouvement brownien, μ est une mesure aléatoire de Poisson et K est un processus croissant permettant à la solution de satisfaire une contrainte de la forme

$$h(t, Y_t, Z_t, U_t(e), e) \quad \geq \quad 0. \tag{0.1.2}$$

0.1.1 Représentation de fonctions valeurs de problèmes de contrôle impulsionnel markovien

Dans le premier chapitre, nous nous intéressons au lien entre les EDSRs et une famille d'équations aux dérivées partielles appelées inéquations quasi-variationnelles. Ces équations aux dérivées partielles prennent la forme suivante :

$$\min\Big[- \frac{\partial v}{\partial t} - \mathcal{L}v - f \ , \ \inf_{e \in E} \{ v - \mathcal{H}^e v \} \Big] \ = \ 0, \ \text{sur } [0, T) \times \mathbb{R}^d, \ \ v(T, .) \ = \ g(.), \ \text{sur } \mathbb{R}^d,$$

avec $\mathcal{L}$ opérateur local du second ordre :

$$\mathcal{L}v(t,x) \;=\; b(x).D_x v(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x)D_x^2 v(t,x)) \;,$$

et $\mathcal{H}$ opérateur non local défini par :

$$\mathcal{H}^e v(t,x) \;=\; v(t,x+\gamma(x,e)) + c(x,e) \;.$$

Les inéquations quasi-variationnelles interviennent en optimisation stochastique dans le cas du contrôle impulsionel (voir par exemple [8]) dont la fonction valeur est définie par

$$v(t,x) \;=\; \sup_{\alpha=(\tau_i,\xi_i)_{i\geq 0}} \mathbb{E}\Big[g(X_T^{t,x,\alpha}) + \int_t^T f(X_s^{t,x,\alpha})ds + \sum_{t<\tau_i\leq T} c(X_{\tau_i^-}^{t,x,\alpha},\xi_i)\Big].$$

Le contrôle $\alpha=(\tau_i,\xi_i)_{i\geq 0}$ est composé d'une suite de temps d'arrêts $(\tau_i)_{i\geq 0}$, représentant les instants auxquels l'agent décide d'intervenir, et d'une suite de variables aléatoires $(\xi_i)_{i\geq 0}$, représentant l'amplitude de l'intervention de l'agent à chacun des instants τ_i. Pour un tel contrôle $\alpha=(\tau_i,\xi_i)_{i\geq 0}$, le processus d'état contrôlé $X^{t,x,\alpha}$ partant de x à l'instant t est donné par

$$X_s^{t,x,\alpha} \;=\; x + \int_t^s b(X_u^{t,x,\alpha})du + \int_t^s \sigma(X_u^{t,x,\alpha})dW_u + \sum_{t<\tau_i\leq s} \gamma(X_{\tau_i^-}^{t,x,\alpha},\xi_i)$$

Une approche possible, pour comprendre le lien entre inéquations quasi-variationnelles et EDSRs, est de remarquer qu'en introduisant une diffusion à sauts X suivant la dynamique :

$$dX_t \;=\; b(X_t)dt + \sigma(X_t)dW_t + \int_E \gamma(X_{t^-},e)\mu(de,dt),$$

et une solution régulière v de l'inéquation quasi-variationnelle (0.1.3), alors le processus $v(t,X_t)$ est, par application de la formule d'Itô, solution d'une EDSR avec contrainte sur le terme de saut :

$$\begin{aligned} v(t,X_t) &= g(X_T) + \int_t^T f(s,X_s)ds - \int_t^T Z_s.dW_s - \int_t^T\int_E U_s(e)\mu(de,ds) + K_T - K_t, \\ U_t(e) &= v(t,X_t+\gamma(X_t,e)) - v(t,X_t) \geq c(X_t,e), \end{aligned}$$

Ce lien entre inéquations quasi-variationnelles et EDSRs contraintes est également suggéré dans [10]. Il identifie, sous des hypothèses assez générales, la fonction valeur d'un problème de cible stochastique avec processus à saut non contrôlé avec la fonction valeur d'un problème de *switching* optimal. Cette fonction valeur du problème de cible stochastique correspond à la solution minimale de notre EDSR contrainte dans le cas d'un générateur ne dépendant pas de l'inconnue (Y,Z,U).

Nous considérons dans ce chapitre une version markovienne des EDSRs contraintes :

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s,Y_s,Z_s)ds - \int_t^T Z_s.dW_s \\ &\quad - \int_t^T\int_E [U_s(e) - c(X_{s^-},Y_{s^-},Z_s,e)]\mu(de,ds) + K_T - K_t, \qquad (0.1.3) \end{aligned}$$

où la contrainte porte uniquement sur le terme de saut :

$$h(U_t(e), e) \quad \geq \quad 0 \ , \tag{0.1.4}$$

X étant une diffusion à sauts markovienne définie par :

$$dX_t \quad = \quad b(X_t)dt + \sigma(X_t)dW_t + \int_E \gamma(X_{t^-}, e)\mu(de, dt) \ .$$

Notons que par changement de variable $V_s(e) = U_s(e) - c(X_{s^-}, Y_{s^-}, Z_s, e)$, cette EDSR apparaît comme un cas particulier de (0.1.1)-(0.1.2). Sous des hypothèses assez générales, nous prouvons l'existence et l'unicité de solutions minimales à de telles équations. Nous introduisons pour cela, en s'inspirant de la littérature sur les EDSRs soumises à des contraintes (voir par exemple [29], [24]), une suite d'EDSRs, dites pénalisées, où le terme à annuler est pénalisé par un facteur explosif. Dans notre cas ces EDSRs pénalisées prennent la forme suivante :

$$\begin{aligned} Y_t^n \quad &= \quad g(X_T) + \int_t^T f(X_s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n.dW_s \\ & \quad - \int_t^T \int_E [U_s^n(e) - c(X_{s^-}, Y_{s^-}, Z_s, e)]\mu(de, ds) + K_T^n - K_t^n, \end{aligned}$$

où le terme K^n est explicité :

$$K_t^n \quad = \quad n \int_0^t \max\{-h(U_s^n(e), e), 0\}\lambda(de)ds,$$

λ étant le compensateur de la mesure μ. Nous montrons que la suite $(Y^n, Z^n, U^n)_n$ converge vers la solution minimale de (0.1.3)-(0.1.4). La difficulté principale pour le passage à la limite est la présence du terme non linéaire f. Nous résolvons cette difficulté en prouvant une généralisation du théorème de limite monotone de Peng [64], au cas des EDSRs à sauts.

Cette suite d'équations pénalisées nous permet ensuite de faire le lien entre la solution minimale de notre EDSR contrainte et les solutions, au sens de la viscosité, d'inéquations quasi-variationnelles générales de la forme

$$\min\Big[- \frac{\partial v}{\partial t} - \mathcal{L}v - f(., v, \sigma^\intercal D_x v) \ , \ \inf_{e \in E} h(\mathcal{H}^e v - v, e)\Big] \quad = \quad 0, \ \text{sur } [0, T) \times \mathbb{R}^d, \tag{0.1.5}$$

avec un opérateur non local $\mathcal{H}$ plus général :

$$\mathcal{H}^e v(t, x) \quad = \quad v(t, x + \gamma(x, e)) + c(x, v(t, x), \sigma^\intercal D_x v(t, x), e).$$

Nous utilisons le lien entre EDSRs à sauts et équations aux dérivées partielles intégrales donné par Barles, Buckdhan et Pardoux [5] pour lier notre EDSR pénalisée à l'équation aux dérivées partielle suivante :

$$-\frac{\partial v}{\partial t} - \mathcal{L}v - f(., v, \sigma^\intercal D_x v) - n \int_E \max\{-h(\mathcal{H}^e v - v, e), 0\}\lambda(de) \quad = \quad 0,$$

sur $[0,T)\times\mathbb{R}^d$. L'utilisation d'argument analytiques nous permettent alors de passer les propriétés de viscosité à la limite pour obtenir une solution à l'inéquation quasi-variationnelle.

Pour compléter l'équation aux dérivées partielles (0.1.5), nous introduisons une condition en T relaxée :

$$\min\Big[v(T,.)-g(.)\ ,\ \inf_{e\in E} h(\mathcal{H}^e v(T^-,.)-v(T^-,.),e)\Big] \ = \ 0,\ \text{sur } \mathbb{R}^d. \qquad (0.1.6)$$

La propriété de viscosité pour cette condition terminale est obtenue par une caractérisation dynamique de la minimalité de la solution à l'EDSR contrainte.

Enfin sous des hypothèses de convexité, nous montrons un résultat d'unicité pour les solutions d'inéquations quasi-variationnelles (0.1.5)-(0.1.6) , ce qui nous permet de lier la solution de l'EDSR contrainte à la fonction valeur du problème de contrôle impulsionnel.

Ce chapitre est tiré d'un article rédigé conjointement avec Jin Ma, Huyên Pham et Jianfeng Zhang [46]. Cet article a été accepté pour publication dans la revue *The Annals of Probability.*

0.1.2 Représentation de fonctions valeurs de problèmes de *switching* optimal et lien entre EDSRs contraintes à sauts et EDSRs à réflexions obliques

Dans ce chapitre, nous étudions la représentation, par EDSRs contraintes à sauts, des solutions de problèmes de *switching* optimal et de systèmes d'inéquations variationnelles.

Récemment, Hu et Tang [44] ont obtenu une représentation stochastique pour les systèmes d'inéquations variationnelles en introduisant la notion d'EDSRs multi-dimensionnelles à réflexions obliques. Ce type d'EDSR, qui a ensuite été généralisé par Hamadène et Zhang [41], prend la forme générale suivante :

$$\begin{cases} Y_t^i = \xi^i + \int_t^T \psi_i(s, Y_s^1,\dots,Y_s^m, Z_s^i)ds - \int_t^T \langle Z_s^i, dW_s\rangle + K_T^i - K_t^i\ , \\ Y_t^i \geq \max_{j\in A_i} h_{i,j}(t, Y_t^j)\ , \\ \int_0^T [Y_t^i - \max_{j\in A_i}\{h_{i,j}(t,Y_t^j)\}]dK_t^i = 0. \end{cases} \qquad (0.1.7)$$

Nous lions cette classe d'EDSRs aux EDSRs du type (0.1.1)-(0.1.2) lorsque le support E de la mesure aléatoire μ est l'ensemble des indices $\mathcal{I} := \{1,\dots,d\}$, d étant la dimension de (0.1.7). Nous montrons que le processus $(Y_t^{I_t})_{t\in[0,T]}$ est solution minimale de l'EDSR contrainte à sauts suivante :

$$\begin{aligned} \tilde{Y}_t \ &= \ \xi^{I_T} + \int_t^T \psi_{I_s}(s, \tilde{Y}_s + \tilde{U}_s(1)\mathbf{1}_{I_s\neq 1},\dots,\tilde{Y}_s + \tilde{U}_s(m)\mathbf{1}_{I_s\neq m}, \tilde{Z}_s)ds \\ &\quad + \tilde{K}_T - \tilde{K}_t - \int_t^T \tilde{Z}_s.dW_s - \int_t^T\int_{\mathcal{I}} \tilde{U}_s(i)\mu(ds,di), \end{aligned}$$

avec

$$\mathbf{1}_{A_{I_{t^-}}}(i)\Big[\tilde{Y}_{t^-} - h_{I_{t^-},i}(t,\tilde{Y}_{t^-} + \tilde{U}_t(i))\Big] \quad \geq \quad 0,$$

le processus I désignant l'indice chargé par la mesure μ à chaque instant. Ce lien est obtenu par identification des équations pénalisées associées.

Une fois ce lien fait, nous nous intéressons à la représentation de problèmes de *switching* dans le cas non markovien. Le problème de *switching* optimal consiste à maximiser la quantité

$$\mathbb{E}\Big[g(X_T^\alpha,\alpha_T) + \int_0^T f(t,X_t^\alpha,\alpha_t)dt - \sum_{\tau_k\leq T} c(\tau_k,\xi_{k-1},\xi_k)\Big]$$

suivant le contrôle α qui, comme dans le cas du contrôle impulsionnel, est constitué d'une suite de temps d'arrêts $(\tau_k)_{k\geq 1}$ et d'une suite de variables aléatoires $(\xi_k)_{k\geq 1}$ adaptées aux temps d'arrêts. La diffusion X^α ayant une dynamique contrôlée de la manière suivante :

$$dX_t^\alpha \quad = \quad b(X_t^\alpha,\alpha_t)dt + \sigma(X_t^\alpha,\alpha_t)dW_t\ , \tag{0.1.8}$$

avec $\alpha_t = \xi_k$ pour $t \in [\tau_k,\tau_{k+1})$. Une première représentation, à l'aide d'une transformation de Girsanov, a été obtenue par Hu et Tang [44] dans le cas où la diffusion sous jacente du problème n'est que partiellement contrôlée sous la forme :

$$dX_t^\alpha \quad = \quad \sigma(X_t^\alpha)\Big[b(X_t^\alpha,\alpha_t)dt + dW_t\Big]\ .$$

Nous généralisons ce résultat en montrant que les EDSRs du type (0.1.1)-(0.1.2) donnent une représentation des processus valeur de problèmes de switching dans le cas d'une diffusion totalement controlée suivant l'équation (0.1.2).

Nous introduisons pour cela une famille d'EDSRs à réflexions obliques paramétrée par un temps d'arrêt et une variable aléatoire. Ce paramètre représente la condition initiale de la diffusion sous-jacente de l'EDSR. En utilisant les représentations des enveloppes de Snell et des arrêts optimaux par EDSRs simplement réfléchies données par [29], nous caractérisons la stratégie optimale comme temps de réflexion associés à une suite d'EDSRs et montrons que le processus valeur associé est bien solution d'une EDSR du type (0.1.1)-(0.1.2).

Ce chapitre a donné lieu à un article rédigé conjointement avec Romuald Elie [32].

0.2 Seconde Partie : Approximation des solutions d'EDSRs à sauts contraints et d'EDSRs à réflexions obliques

Nous étudions dans cette partie deux méthodes probabilistes d'approximation de solutions de systèmes d'inéquations variationnelles. La première utilise la représentation par EDSRs contraintes à sauts et nous donne un algorithme fondé sur la résolution d'EDSRs pénalisées associées. La seconde utilise la notion d'EDSR à réflexions obliques discrètes. Dans ce dernier cas, nous obtenons une vitesse de convergence de l'approximation vers la solution continûment réfléchie.

0.2.1 Représentation et approximation probabiliste pour des systèmes couplés d'inéquations variationnelles

Nous nous intéressons dans ce chapitre à la représentation probabiliste de solutions de systèmes d'inéquations variationnelles afin de mettre en place une procédure d'approximation numérique de ces solutions.

Nous considérons alors la solution de l'EDSR (0.1.1)-(0.1.2) dans le cas où :

- le support E de la mesure aléatoire μ est l'ensemble $\mathcal{I} := \{1, \ldots, d\}$,
- les coefficients sont markoviens dirigés par un processus de diffusion-transmutation (I, X^I) introduit par Pardoux Pradeilles et Rao [63] pour prendre en compte les changements de régime des opérateurs différentiels intervenant dans les systèmes d'EDPs associés.

L'EDSR étudiée prends alors la forme suivante :

$$\begin{aligned} Y_t &= g(X_T, I_T) + \int_t^T f_{I_s}(X_s, Y_s + U_s(1), \ldots, Y_s + U_s(d), Z_s)ds \\ &\quad - \int_t^T Z_s.dW_s - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(di, ds) + K_T - K_t \;, \end{aligned} \tag{0.2.9}$$

la solution devant satisfaire une contrainte de la forme :

$$h_{I_{t^-},j}(t, X_t^I, Y_t, Y_t + U_t(j), Z_t) \quad \geq \quad 0 \;.$$

Le processus de diffusion-transmutation (I, X^I) dirigeant cette EDSR étant défini par :

$$\begin{aligned} dI_t &= \int_{i \in \mathcal{I}} (i - I_{t^-})\mu(di, dt) \;, \\ dX_t^I &= b(X_t^I, I_t)dt + \sigma(X_t^I, I_t)dW_t \;. \end{aligned}$$

Nous montrons que la solution Y de l'EDSR (0.1.1)-(0.1.2) constitue une représentation de type Feynman-Kac pour des solutions de systèmes d'inéquations variationnelles de la forme :

$$\min\Big[- \frac{\partial v_i}{\partial t} - \mathcal{L}^i v_i - f_i(\cdot, v_1, \ldots, v_d, \sigma^\intercal D_x v_i) \;,\; \min_{j \in E} h_{i,j}(\cdot, v_i, v_j, \sigma^\intercal D_x v_i) \Big] \quad = \quad 0,$$

sur $[0, T) \times \mathbb{R}^d$ pour $i \in \{1, \ldots, d\}$. $\mathcal{L}^i$ étant l'opérateur local du second ordre défini par :

$$\mathcal{L}^i \varphi(t, x) \quad = \quad b(x, i).D_x \varphi(t, x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x, i) D_x^2 \varphi(t, x)) \;.$$

Ce lien avec la solution de l'EDP nous permet de mieux comprendre le rôle du processus K dans l'equation (0.2.9). Nous obtenons en effet une condition de minimalité, similaire à celle obtenue dans le cas réfléchi (voir [29]), lorsque la fonction h ne dépend pas de la variable Z :

$$\int_0^T \min_j \{h_{I_{t^-},j}(t, X_t^I, Y_{t^-}, Y_{t^-} + U_t(j))\} dK_t \quad = \quad 0.$$

L'approximation de l'EDSR (0.2.9) se fait alors par discrétisation de l'EDSR pénalisée associée en utilisant les résultats de convergence de Bouchard et Elie [12].

Ce chapitre a donné lieu à une noté rédigée conjointement avec Romuald Elie [33].

0.2.2 Discrétisation des EDSR multi-dimentionnelles à réflexions obliques

Dans ce chapitre, nous étudions l'approximation d'EDSRs à réflexions obliques de la forme :

$$\begin{cases} \dot{Y}_t^i = g^i(X_T) + \int_t^T f_i(X_s, \dot{Y}_s^i, \dot{Z}_s^i)ds - \int_t^T \dot{Z}_s^i.dW_s + \dot{K}_T^i - \dot{K}_t^i \ , \\ \dot{Y}_t^i \geq \max_{j\neq i}\{\dot{Y}_t^j - c_{i,j}(X_t)\} \ , \\ \int_0^T [\dot{Y}_t^i - \max_{j\neq i}\{\dot{Y}_t^j - c_{i,j}(X_t)\}] d\dot{K}_t^i = 0 \ , \end{cases} \tag{0.2.10}$$

avec X la diffusion définie par :

$$dX_t \quad = \quad b(X_t)dt + \sigma(X_t)dW_t \ . \tag{0.2.11}$$

Nous introduisons pour une grille de réflexion donnée $\Re := \{r_0 = 0, \ldots, r_\kappa = T\}$, l'EDSR à réflexions obliques discrètes, étudiée pour les EDSRs unidimensionnelles réfléchies par Bouchard et Chassagneux [11] dans le cas d'une barrière simple et par Chassagneux [22] dans le cas d'une double barrière :

$$\begin{cases} \widetilde{Y}_t = g(X_T) + \int_t^T f(X_s, \widetilde{Y}_s, \widetilde{Z}_s)ds - \int_t^T \widetilde{Z}_s.dW_s + \widetilde{K}_T - \widetilde{K}_t \ , \\ \widetilde{K}_t = \sum_{t\leq T} \Delta\widetilde{K}_t \text{ et } \Delta\widetilde{K}_t = \widetilde{Y}_t - Y_t \ , \\ Y_t = \widetilde{Y}_t \mathbf{1}_{t\notin\Re} + \mathcal{P}(X_t, \widetilde{Y}_t)\mathbf{1}_{t\in\Re} \ , \end{cases} \tag{0.2.12}$$

avec $\mathcal{P}(x, .)$ opérateur de projection oblique :

$$\mathcal{P}(x,y) \quad = \quad \Big(\max_{1\leq j\leq d} \{y^i - c_{i,j}(x)\} \Big)_{1\leq i\leq d} \ .$$

Nous considérons alors le schéma associé à cette EDSR discrètement réfléchie :

$$\begin{cases} \bar{Z}_{t_i}^\pi & := & (t_{i+1} - t_i)^{-1} \mathbb{E}_{t_i}[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i})'] \ , \\ \widetilde{Y}_{t_i}^\pi & := & \mathbb{E}_{t_i}[Y_{t_{i+1}}^\pi] + (t_{i+1} - t_i) f(X_{t_{i+1}}^\pi, \widetilde{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \ , \\ Y_{t_i}^\pi & := & \widetilde{Y}_{t_i}^\pi \mathbf{1}_{t_i\notin\Re} + \mathcal{P}(X_{t_{i+1}}^\pi, \widetilde{Y}_{t_i}^\pi)\mathbf{1}_{t_i\in\Re} \ , \end{cases} \tag{0.2.13}$$

pour lequel nous montrons, par des techniques présentes dans la littérature ([56, 11, 21]), la convergence vers la solution de (0.2.12). Cependant, la spécificité de l'opérateur de projection oblique constitue une difficulté pour l'application des méthodes classiques permettant d'obtenir une vitesse de convergence.

Nous utilisons alors l'approche de Hu et Tang [44], consistant à voir les solutions d'EDSR à réflexions obliques comme processus valeur de problème de *switching* optimal d'EDSRs unidimensionnelles. Cette approche nous permet, via des théorèmes de comparaison pour les EDSRs d'obtenir un contrôle sur l'écart entre $\dot{Y}$ et $\widetilde{Y}$. Nous montrons alors, dans le cas où le générateur f ne dépends pas de la variable Z, la convergence, pour la composante Y, du schéma (0.2.13) vers l'EDSR discrètement réfléchie (0.2.12) à une vitesse $|\pi|^{\frac{1}{2}}$, lorsque le pas de discrétisation $|\pi|$ tends vers 0.

Sous une hypothèse de bornitude du générateur f en la variable Z nous montrons, toujours en utilisant cette interprétation, la convergence lorsque le pas de la grille de réflexion $|\Re|$ tends vers 0, de l'EDSR discrètement réfléchie (0.2.12) vers l'EDSR (0.2.10), à la vitesse $|\Re|^{\frac{1-\varepsilon}{2}}$ sur les points de la grille, pour tout $\varepsilon > 0$. Dans le cas particulier de fonctions de coûts constantes nous obtenons une vitesse de $|\Re|^{\frac{1}{2}}$ pour cette même convergence.

La composition de ces deux résultats dans le cas $\Re = \pi$, nous donne alors une majoration de l'écart entre les composantes Y de l'EDSR (0.2.10) et du schéma (0.2.13).

Ce chapitre est tiré d'un travail réalisé en collaboration avec Jean-François Chassagneux et Romuald Elie.

0.3 Troisième Partie : Un modèle de liquidation optimale de portefeuille avec coût et risque d'exécution

Comprendre le fonctionnement des marchés financiers est un enjeu fondamental pour les praticiens de la finance. Une question importante que se posent les intervenants est comment liquider une position importante en un certain actif. Un dilemme se pose alors. En échangeant rapidement, l'intervenant est soumis à des coûts élevés dus à l'épuisement du carnet d'ordre. Il est donc préférable d'espacer ses interventions, mais alors l'intervenant est soumis aux variations de cours du marché. Il y a récemment eu un regain d'intérêt dans la littérature pour ces effets de liquidité qui a donné lieu à de nombreux travaux prenant en compte ces impacts de prix dont Bertsimas et Lo [9], Almgren et Criss [1], Bank et Baum [6], Cetin, Jarrow et Protter [19], Obizhaeva et Wang [58], He et Mamayski [42], Schied et Schöneborn [74], Ly Vath, Mnif et Pham [53], Rogers et Singh [72], et Cetin, Soner et Touzi [20].

Il existe principalement deux types de formulations pour les problèmes de gestion de portefeuille : la modélisation en temps continu d'une part et en temps discret d'autre part.

Les modèles en temps continu ne sont pas très réalistes mais restent couramment utilisés, notamment en raison de l'efficacité du calcul stochastique pour leurs résolutions.

Dans le cas des modélisations en temps discret, on peut distinguer trois types de travaux :

- temps d'exercices à dates déterministes fixées (voir [9]),
- temps d'exercices exogènes et aléatoires donnés par exemple par une mesure de Poisson (voir [69], [7]),

– temps d'exercices discrets choisis de manière optimale par l'investisseur sur un intervalle continu (voir [42] et [53]).

Dans ce dernier cas, il est commun de supposer l'existence de coûts de transaction fixes afin de ne pas avoir de stratégies explosives i.e. accumulation des ordres qui se rapprocheraient d'une stratégie continue. Cependant la présence d'un coût fixe de transaction n'est pas toujours en accord avec la réalité des marchés.

Nous présentons un modèle issu de cette dernière famille et prenant en compte les effets du manque de liquidité des marchés.

Nous considérons un marché financier comportant un actif sans risque au taux d'intérêt $r = 0$ et un actif risqué de processus de prix $(P_t)_t$. Nous considérons un agent détenant une certaine quantité Y_0 de cet actif risqué et cherchant à liquider cette position.

Stratégies d'investissement. Notre agent peut acheter ou vendre cet actif risqué suivant une stratégie discrète $\alpha = (\tau_n, \xi_n)_{n\geq 1} : \tau_1 \leq \cdots \tau_n \leq \cdots \leq T$ représentent les temps d'intervention de l'investisseur, et ξ_n, le nombre d'actifs risqués achetés ou vendus lors de ces interventions.

Coût d'intervention. Lorsque notre agent intervient sur le marché à une date $\tau_n \in [0, T]$, en achetant une quantité ξ_n, il paie alors le montant

$$Q(\xi_n, P_{\tau_n}, \tau_n - \tau_{n-1}) \quad = \quad P_{\tau_n} \xi_n f(\xi_n, \tau_n - \tau_{n-1}) \ ,$$

la fonction f représentant l'impact temporaire de l'intervention de l'investisseur sur le prix. Cet impact dépendant de la quantité échangée et de la fréquence d'intervention $\tau_n - \tau_{n-1}$. Une telle modélisation permet de représenter le manque de liquidité dû à l'épuisement du carnet d'ordre venant soit d'une intervention importante, soit d'une multitude d'interventions de tailles moins importantes mais à des dates très proches. En particulier, l'investisseur ne peut pas réduire l'impact de son intervention en segmentant un ordre important en plusieurs ordres de tailles moindres.

Cette modélisation permet aussi de définir une condition de solvabilité. Nous imposons à la richesse potentielle $L(Z_t, \Theta_t) = L(X_t, Y_t, P_t, \Theta_t)$ de l'investisseur au temps t d'être positive. Cette richesse potentielle étant définie comme la richesse si l'investisseur liquide immédiatement sa position en actif risqué :

$$L(X_t, Y_t, P_t, \Theta_t) \quad = \quad X_t + Y_t P_t f(-Y_t, \Theta_t) \ ,$$

X_t, Y_t et Θ_t étant respectivement la quantité de cash, le nombre d'actifs risqués cumulés et le délais depuis le dernier ordre passé au temps t. Une telle contrainte de solvabilité constitue un atout important dans le contexte actuel de régulation bancaire en vue de limiter les risques systémiques.

Le problème d'investissement. Nous étudions le problème de maximisation de l'espérance d'utilité de la richesse terminale sous contrainte de liquidation de la position en actif risqué,

$Y_T = 0$, et sous contrainte de position positive en l'actif risqué $Y_t \geq 0$ pour $t \in [0, T]$:

$$v(t, z, \theta) = \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}[U(X_T^\alpha)] ,$$

$\mathcal{A}(t, z)$ représentant l'ensemble des contrôles satisfaisant les contraintes.

Un premier résultat important est de montrer que les stratégies (presque) optimales sont finies en dépit de l'absence de coûts fixes de transaction.

Nous montrons ensuite que la fonction valeur de notre problème est solution d'une inéquation quasi-variationnelle :

$$\min\Big[-\frac{\partial v}{\partial t} - \frac{\partial v}{\partial \theta} - \mathcal{L}v, v - \mathcal{H}v \Big] = 0. \tag{0.3.14}$$

Malheureusement, la forme singulière de l'opérateur non local $\mathcal{H}$ de cette inéquation quasi-variationnelle ne permet pas de prouver l'unicité de la solution par la méthode habituelle de construction d'une sur-solution stricte, initialement introduite par Ishii [45].

Nous caractérisons alors cette fonction valeur comme limite d'une suite de fonctions valeurs associées à des problèmes d'investissement identiques mais avec l'ajout d'un coût de transaction convergeant vers 0. Le modèle est identique avec un coût d'échange Q_ε de la forme :

$$Q_\varepsilon(\xi_n, P_{\tau_n}, \tau_n - \tau_{n-1}) = Q(\xi_n, P_{\tau_n}, \tau_n - \tau_{n-1}) - \varepsilon ,$$

et une fonction de liquidation L_ε de la forme :

$$L_\varepsilon(Z_t, \Theta_t) = L(Z_t, \Theta_t) - \varepsilon .$$

Nous considérons alors la suite de fonctions valeurs (v_ε^{ct}) définie par :

$$v_\varepsilon^{ct}(t, z) = \sup_{\alpha \in \mathcal{A}^\varepsilon(t,z)} \mathbb{E}[U(L_\varepsilon(Z_T^\varepsilon, \Theta_T))] ,$$

La présence de ce coût de transaction fixe permet alors de caractériser la fonction valeur approchée v_ε^{ct} comme unique solution de son l'inéquation quasi-variationnelle. Nous montrons ensuite la convergence ponctuelle de v_ε^{ct} vers v lorsque ε tends vers 0.

Enfin une seconde approximation de notre fonction valeur par pénalisation d'utilité est mise en place. Il s'agit du problème d'optimisation :

$$v_\varepsilon^{pu}(t, z) = \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}[U(X_T) - \varepsilon N(\alpha)] ,$$

avec $N(\alpha)$ le nombre d'ordre de la stratégie α. Cette approximation nous permet de montrer que la fonction valeur v est la solution minimale de l'inéquation quasi-variationnelle (0.3.14).

Ce chapitre est issu d'un article rédigé en collaboration avec Huyên Pham [47].

INTRODUCTION AND SUMMARY

This thesis deals with the study of stochastic optimization problems and their applications in financial mathematics.

The first part is devoted to the probabilistic representation, in terms of Backward Stochastic Differential Equations, of optimization problems. Specifically we focus on the impulse control and optimal switching problems.

In the first chapter, we consider the case of the impulse control problem in a Markovian framework. It is well known (see e. g. [8]) that stochastic impulse control is linked to a class of parabolic partial differential equations called Quasi-Variational Inequalities. We interpret these QVIs as problems with given terminal condition whose solutions are submitted to a specific constraint. This gives a suitable framework for the study of BSDEs with constrained jumps. We then introduce these so-called BSDEs with constrained jumps for which we prove, under mild conditions, the existence and uniqueness of minimal solutions. For this purpose we introduce, as classically done in the literature, a sequence of penalized BSDEs. Adapting the approach of Peng [64], we prove the convergence of this sequence of penalized solutions to the minimal constrained solution. We then connect this minimal constrained solution to QVIs. Using the results of [5], we link the penalized solutions to integral PDE. Then using viscosity arguments inspired by [4], we get the viscosity property of the minimal constrained solution using a limit argument. Finaly, we provide a comparison theorem for the studied QVI under some convexity assumptions on the coefficients.

In the second chapter, we study the probabilistic representation of optimal switching problems. Recently, such a representation was obtained by Hu and Tang [44], in terms of BSDEs with oblique reflections. However this representation does not cover the general case of optimal switching: the underlying diffusion is controlled only via its drift term. We provide a stochastic representation for the general problem, in terms of constrained BSDEs with jumps. For this purpose, we introduce a family of BSDEs with oblique reflections indexed by the initial condition of the underlying diffusion. Using the representation of

Snell envelopes and optimal stopping times in terms of one dimentional reflected BSDEs, we provide an optimal switching strategy by jumping from a BSDE to another one in this family. Finaly we link theses two types of BSDEs: constrained BSDEs with jumps and obliquely reflected ones.

In the second part, we study the numerical approximation of BSDEs associated to variational inequalities. In the first chapter we give a representation in terms of constrained BSDEs with jumps, for solutions to systems of variational inequalities. As in Pardoux, Pradeilles and Rao [63], we consider a markovian BSDE driven by a diffusion-transmutation process, to take into account the fact that the differential operators are different at each line of the system. We then link theses constrained BSDEs with jumps to general fully coupled systems of variational inequalities. For this purpose we introduce penalized BSDEs with jumps which are related to integral PDEs according to Barles, Buckdahn and Pardoux [5]. We get the viscosity property using a limit argument. We also provide a numerical approximating procedure, relying on the recent results on time discretization of BSDEs with jumps by Bouchard and Elie [12].

The second chapter deals with the discrete-time approximation of BSDEs with oblique reflections. We first study the discrete-time approximation of discretely reflected BSDEs, introduced in [11, 22] for the one dimensional case, which are classical BSDEs being obliquely projected on a convex set only on a finite time grid $\Re$. We first prove the convergence of the associated discrete-time scheme. Due to the particular form of the oblique projection operator, the classical methods used in the literature cannot be applied to get a rate for the previous convergence. Using the interpretation of the solution of such a BSDE as the value process of an optimal switching problem as in [44], we prove that the component Y of the scheme converges to the discretely reflected BSDE's one at a rate $|\pi|^{\frac{1}{2}}$.

Still using this approach, we prove the convergence of the discretely reflected BSDE to the continuously reflected at a rate $|\Re|^{\frac{1-\varepsilon}{2}}$, for all $\varepsilon > 0$, when $|\Re|$ goes to 0, on the grid points, for the particular case where f does not depend on Z.

Combining these two results, we obtain a rate for the convergence of the scheme to the continuously reflected BSDE.

In the third part, we study a model of optimal liquidation with execution cost and risk. We propose a continuous-time framework taking into account the main liquidity features and risk/cost tradeoff of portfolio execution: there is a bid-ask spread in the limit order book, and temporary market price impact penalizing rapid execution trades. However, in contrast with previous papers (see e.g. [74] or [72]), we do not assume continuous-time trading strategies. We consider instead real trading taking place in discrete-time, and without assuming any ad-hoc fixed transaction cost, in accordance with the practitioner literature. We consider an investor who has to liquidate at maturity, a quantity of risky asset with price process P, and who tries to maximize its terminal expected utility. We then prove that the associated value function is a viscosity solution to a quasi-variational inequality (QVI in short). Unfortunately, the particular form of the nonlocal operator $\mathcal{H}$ involved in the previous QVI does not allow to prove uniqueness by the classical method of Ishii [45].

We then provide two approximations to characterize the value function v. The first one consists in introducing a small fixed transaction cost ε. In this case the associated value function v_ε^{tc} is characterized as the unique viscosity solution of its associated QVI. Then we prove that the sequence $(v_\varepsilon^{tc})_\varepsilon$ converges to v pointwisely as ε goes to zero.

A second approximation consists in penalizing by a small parameter ε the terminal utility for each intervention of the investor. We prove that the sequence of associated value functions converges to v as ε goes to 0. This last approximation allows to prove that v is the smallest viscosity solution to its associated QVI.

Part I

PROBABILISTIC REPRESENTATION OF SEQUENTIAL OPTIMAL STOCHASTIC CONTROL IN CONTINUOUS TIME

Chapter 1

Backward SDEs with constrained jumps and Quasi-Variational Inequalities

Abstract : We consider a class of backward stochastic differential equations (BSDEs) driven by Brownian motion and Poisson random measure, and subject to constraints on the jump component. We prove the existence and uniqueness of the minimal solution for the BSDEs by using a penalization approach. Moreover, we show that under mild conditions the minimal solutions to these constrained BSDEs can be characterized as the unique viscosity solution of quasi-variational inequalities (QVIs), which leads to a probabilistic representation for solutions to QVIs. Such a representation in particular gives a new stochastic formula for value functions of a class of impulse control problems. As a direct consequence, this suggests a numerical scheme for the solution of such QVIs via the simulation of the penalized BSDEs.

Keywords: Backward stochastic differential equation, jump-diffusion process, jump constraints, penalization quasi-variational inequalities, impulse control problems, viscosity solutions.

1.1 Introduction and summary

Consider a parabolic quasi-variational inequality (QVI for short) of the following form:

$$\begin{cases} \min\Big[-\dfrac{\partial v}{\partial t} - \mathcal{L}v - f\ ,\ v - \mathcal{H}v\Big] & = & 0, \quad \text{on } [0,T)\times\mathbb{R}^d, \\ v(T,\cdot) & = & g \quad \text{on } \mathbb{R}^d, \end{cases} \tag{1.1.1}$$

where $\mathcal{L}$ is the second order local operator

$$\mathcal{L}v(t,x) \;=\; \langle b(x), D_x v(t,x)\rangle + \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x)D_x^2 v(t,x)) \tag{1.1.2}$$

and $\mathcal{H}$ is the nonlocal operator

$$\mathcal{H}v(t,x) \;=\; \sup_{e\in E}[v(t,x+\gamma(x,e)) + c(x,e)]. \tag{1.1.3}$$

In the above, $D_x v$ and $D_x^2 v$ are the partial gradient and the Hessian matrix of v with respect to its second variable x, respectively; $^{\intercal}$ stands for the transpose; $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$; $\mathbb{S}^d$ is the set of all symmetric $d\times d$ matrices; and E is some compact subset of $\mathbb{R}^q$.

It is well-known (see, e.g., [8]) that the QVI (1.1.1) is the dynamic programming equation associated to the impulse control problems whose value function is defined by:

$$v(t,x) \;=\; \sup_{\alpha=(\tau_i,\xi_i)_i} \mathbb{E}\Big[g(X_T^{t,x,\alpha}) + \int_t^T f(X_s^{t,x,\alpha})ds + \sum_{t<\tau_i\le T} c(X_{\tau_i^-}^{t,x,\alpha},\xi_i)\Big]. \tag{1.1.4}$$

More precisely, given a filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$ where $\mathbb{F}=\{\mathcal{F}_t\}_t$, we define an impulse control α as a double sequence $(\tau_i,\xi_i)_i$ in which $\{\tau_i\}$ is an increasing sequence of $\mathbb{F}$-stopping times, and each ξ_i is an $\mathcal{F}_{\tau_i}$-measurable random variable taking values in E. For each impulse control $\alpha=(\tau_i,\xi_i)_i$, the controlled dynamics starting from x at time t, denoted by $X^{t,x,\alpha}$, is a càdlàg process satisfying the following SDE:

$$X_s^{t,x,\alpha} = x + \int_t^s b(X_u^{t,x,\alpha})du + \int_t^s \sigma(X_u^{t,x,\alpha})dW_u + \sum_{t<\tau_i\le s}\gamma(X_{\tau_i^-}^{t,x,\alpha},\xi_i), \tag{1.1.5}$$

where W is a d-dimensional $\mathbb{F}$-Brownian motion. In other words, the controlled process $X^{t,x,\alpha}$ evolves according to a diffusion process between two successive intervention times τ_i and τ_{i+1}, and at each decided intervention time τ_i, the process jumps with size $\Delta X_{\tau_i}^{t,x,\alpha} := X_{\tau_i}^{t,x,\alpha} - X_{\tau_i^-}^{t,x,\alpha} = \gamma(X_{\tau_i^-}^{t,x,\alpha},\xi_i)$.

We note that the impulse control problem (1.1.4) may be viewed as a sequence of optimal stopping problems combined with jumps in state due to impulse values. Moreover, the QVI (1.1.1) is the infinitesimal derivation of the dynamic programming principle, which means that at each time, the controller may decide either to do nothing and let the state process diffuse, or to make an intervention on the system via some impulse value. The former is characterized by the linear PDE in (1.1.1), while the latter is expressed by the obstacle (or reflected) part in (1.1.1). From the theoretical and numerical point of view, the main difficulty of the QVI (1.1.1) lies in that the obstacle contains the solution itself, and it is nonlocal (see (1.1.3)) due to the jumps induced by the impulse control. These features make the classical approach of numerically solving such impulse control problems particular challenging.

An alternative method to attack the QVI (1.1.1) is to find the probabilistic representation of the solution using the Backward Stochastic Differential Equations (BSDEs), namely the so-called nonlinear Feynman-Kac formula. One can then hope to use such a representation to derive a direct numerical procedure for the solution of QVIs, whence the impulse control problems. The idea is the following. We consider a Poisson random measure $\mu(dt, de)$ on $\mathbb{R}_+ \times E$ associated to a marked point process $(T_i, \zeta_i)_i$. Assume that μ is independent of W and has intensity $\lambda(de)dt$, where λ is a finite measure on E. Consider a (uncontrolled) jump-diffusion process

$$X_s = X_0 + \int_0^s b(X_u)du + \int_0^s \sigma(X_u)dW_u + \sum_{T_i \le s} \gamma(X_{T_i^-}, \zeta_i). \tag{1.1.6}$$

Assume that v is a “smooth" solution to (1.1.1), and define $Y_t = v(t, X_t)$. Then, by Itô's formula we have

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle \\ &\quad - \int_t^T \int_E (U_s(e) - c(X_{s^-}, e))\mu(ds, de), \end{aligned} \tag{1.1.7}$$

where $Z_t = \sigma^\intercal(X_{t^-})D_x v(t, X_{t^-})$, $U_t(e) = v(t, X_{t^-} + \gamma(X_{t^-}, e)) - v(t, X_{t^-}) + c(X_{t^-}, e)$, and $K_t = \int_0^t (-\frac{\partial v}{\partial t} - \mathcal{L}v - f)(s, X_s)ds$. Since v satisfies (1.1.1), we see that K is a continuous (hence predictable), nondecreasing process, and U satisfies the constraint:

$$- U_t(e) \ \ge \ 0, \tag{1.1.8}$$

The idea is then to view (1.1.7) and (1.1.8) as a BSDE with jump constraints, and we expect to retrieve $v(t, X_t)$ by solving the “minimal" solution (Y, Z, U, K) to this constrained BSDE.

We can also look at the BSDE above slightly differently. Let us denote $d\bar{K}_t = dK_t - \int_E U_s(e)\mu(dt, de)$, $t \ge 0$. Then $\bar{K}$ is still a nondecreasing process, and the equation (1.1.7) can now be rewritten as

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s)ds + \int_t^T \int_E c(X_{s^-}, e)\mu(ds, de) \\ &\quad - \int_t^T \langle Z_s, dW_s \rangle + \bar{K}_T - \bar{K}_t. \end{aligned} \tag{1.1.9}$$

We shall prove that $v(t, X_t)$ can also be retrieved by looking at the minimal solution $(Y, Z, \bar{K})$ to this BSDE. In fact, the following relation holds (assuming $t = 0$):

$$\begin{aligned} v(0, X_0) &= \inf\{y \in \mathbb{R} \ : \exists Z, \ \ y + \int_0^T \langle Z_s, dW_s\rangle \geq \\ & \qquad g(X_T) + \int_0^T f(X_s)ds + \int_0^T \int_E c(X_{s^-}, e)\mu(ds, de)\}. \end{aligned} \tag{1.1.10}$$

Notice that (1.1.10) also has a financial interpretation. That is, $v(0, x)$ is the minimal capital allowing to superhedge the payoff $\Pi_T(X) = g(X_T) + \int_0^T f(X_s)ds + \int_0^T c(X_{s^-}, e)\mu(ds, de)$ by trading only the asset W. Here, the market is obviously incomplete, since the jump part of the underlying asset X is not hedgeable. This connection between the impulse control problem (1.1.4) and the stochastic target problem defined by the r.h.s. of (1.1.10) was originally proved in Bouchard [10].

Inspired by the above discussion, we now introduce the following general BSDE:

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle \\ & \quad - \int_t^T \int_E (U_s(e) - c(X_{s^-}, Y_{s^-}, Z_s, e))\mu(ds, de), \quad 0 \leq t \leq T, \end{aligned} \tag{1.1.11}$$

with constraints on the jump component in the form:

$$h(U_t(e)) \ \geq \ 0, \qquad \forall e \in E, \ 0 \leq t \leq T, \tag{1.1.12}$$

where h is a given nonincreasing function. The solution to the BSDE is a quadruple (Y, Z, U, K) where, besides the usual component (Y, Z, U), the fourth component K is a nondecreasing, càdlàg, adapted process, null at zero, which makes the constraint (1.1.12) possible. We note that without the constraint (1.1.12), the BSDE with $K = 0$ was studied by Tang and Li [77] and Barles, Buckdahn and Pardoux [5]. However, with the presence of the constraint, we may not have the uniqueness of the solution. We thus look only for the minimal solution (Y, Z, U, K), in the sense that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ satisfying (1.1.11)-(1.1.12), it must hold that $Y \leq \tilde{Y}$. Clearly, this BSDE is a generalized version of (1.1.7)-(1.1.8), where the functions f and c are independent of y and z, and $h(u) = -u$.

We can also consider the counterpart of (1.1.9), namely finding the minimal solution (Y, Z, K) of the BSDE:

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds + \int_t^T \int_E c(X_{s^-}, Y_{s^-}, Z_s, e)\mu(ds, de) \\ & \qquad - \int_t^T \langle Z_s, dW_s\rangle + K_T - K_t, \qquad 0 \leq t \leq T. \end{aligned} \tag{1.1.13}$$

It is then conceivable, as we shall prove, that this problem is a special case of (1.1.11)-(1.1.12) with $h(u) = -u$.

It is worth noting that if the generator f and the cost function c do not depend on y, z, which we refer to as the impulse control case, the existence of a minimal solution to the constrained BSDEs (1.1.7)-(1.1.8) may be directly obtained by supermartingale decomposition method in the spirit of El Karoui and Quenez [30] for the dual representation of the super-replication cost of $\Pi_T(X)$. In fact, the results could be extended easily to the case where f is linear in z, via a simple application of the Girsanov transformation. In our general case, however, we shall follow a penalization method, as was done in El Karoui et al. [29]. Namely, we construct a suitable sequence (Y^n, Z^n, U^n, K^n) of BSDEs with jumps, and prove that it converges to the minimal solution that we are looking for. This is achieved as follows. We first show the convergence of the sequence (Y^n) by relying on comparison results for BSDEs with jumps, see [73]. The proof of convergence of the components (Z^n, U^n, K^n) is more delicate, and is obtained by using a weak compactness argument due to Peng [64].

Our next task of this paper is to relate the minimal solution to the BSDE with constrained jumps to the viscosity solutions to the following general QVI:

$$\min\Big[-\frac{\partial v}{\partial t} - \mathcal{L}v - f(\cdot, v, \sigma^{\intercal} D_x v)\ ,\ h(\mathcal{H}v - v)\Big] \quad = \quad 0, \tag{1.1.14}$$

where $\mathcal{H}$ is the nonlocal semilinear operator

$$\mathcal{H}v(t,x) \quad = \quad \sup_{e \in E}\,[v(t, x + \gamma(x,e)) + c(x, v(t,x), \sigma^{\intercal}(x) D_x v(t,x), e)].$$

Under suitable assumptions, we shall also prove the uniqueness of the viscosity solution, leading to a new probabilistic representation for this parabolic QVI.

We should point out that BSDEs with constraints have been studied by many authors. For example, El Karoui et al. [29] studied the reflected BSDEs, in which the component Y is forced to stay above a given obstacle; Buckdahn and Hu [15, 16] followed by Cvitanic, Karatzas and Soner [24] considered the case where the constraints are imposed on the component Z. Recently Peng [64] (see also [65]) studied the the general case where constraints are given on both Y and Z, which relates these constrained BSDEs to variational inequalities. The main feature of this work is to consider constraints on the jump component (U) of the solution, and to relate these jump-constrained BSDEs to quasi-variational inequalities. On the other hand, the classical approach in the theory and numerical approximation of impulse control problems and QVIs is to consider them as obstacle problems and iterated optimal stopping problems. However, our penalization procedure for jump-constrained BSDEs suggests a non-iterative approximation scheme for QVIs, based on the simulation of the BSDEs, which, to our best knowledge, is new.

The rest of the paper is organized as follows: In Section 2 we give a detailed formulation of BSDEs with constrained jumps, and show how it includes problem (1.1.13) as special case. Moreover, in the special case of impulse control, we directly construct and show the existence of a minimal solution. In Section 3 we develop the penalization approach for studying the existence of a minimal solution to our constrained BSDE for general f, c, and h. We show in Section 4 that the minimal solution to this constrained BSDE provides a probabilistic representation for the unique viscosity solution to a parabolic QVI. Finally, in Section 5 we provide some examples of sufficient conditions under which our general assumptions are satisfied.

1.2 BSDEs with constrained jumps

1.2.1 General formulation

Throughout this paper we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space on which are defined a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$, and a Poisson random measure μ on $\mathbb{R}_+ \times E$, where E is a compact set of $\mathbb{R}^q$, endowed with its Borel field $\mathcal{E}$. We assume that the Poisson random measure μ is independent of W, and has the intensity measure $\lambda(de)dt$ for some finite measure λ on $(E, \mathcal{E})$. We set $\tilde{\mu}(dt, de) = \mu(dt, de) - \lambda(de)dt$, the compensated measure associated to μ; and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the augmentation of the natural filtration generated by W and μ, and by $\mathcal{P}$ the σ-algebra of predictable subsets of $\Omega \times [0, T]$.

Given Lipschitz functions $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, and a measurable map $\gamma : \mathbb{R}^d \times E \to \mathbb{R}^d$, satisfying for some positive constants C and k_γ,

$$\sup_{e\in E} |\gamma(x, e)| \;\leq\; C, \quad \text{and} \quad \sup_{e\in E} |\gamma(x, e) - \gamma(x', e)| \;\leq\; k_\gamma |x - x'|, \quad x, x' \in \mathbb{R}^d,$$

we consider the forward SDE:

$$dX_s \;=\; b(X_s)ds + \sigma(X_s)dW_s + \int_E \gamma(X_{s^-}, e)\mu(ds, de). \tag{1.2.1}$$

Existence and uniqueness of (1.2.1) given an initial condition $X_0 \in \mathbb{R}^d$, is well-known under the above assumptions, and for any $0 \leq T < \infty$, we have the standard estimate

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t|^2\Big] \;<\; \infty. \tag{1.2.2}$$

In what follows we fix a finite time duration $[0, T]$. Let us introduce some additional notations. We denote by

- $\mathcal{S}^\mathbf{2}$ the set of real-valued càdlàg adapted processes $Y = (Y_t)_{0\leq t\leq T}$ such that $\|Y\|_{\mathcal{S}^\mathbf{2}} := \Big(\mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big]\Big)^{\frac{1}{2}} \;<\; \infty$.
- $\mathbf{L^p(0, T)}$, $p \geq 1$, the set of real-valued processes $(\phi_t)_{0\leq t\leq T}$ such that $\mathbb{E}\Big[\int_0^T |\phi_t|^p dt\Big] < \infty$; and $\mathbf{L^p_{\mathbb{F}}(0, T)}$ is the subset of $\mathbf{L^p(0, T)}$ consisting of adapted processes.
- $\mathbf{L^p(W)}$, $p \geq 1$, the set of $\mathbb{R}^d$-valued $\mathcal{P}$-measurable processes $Z = (Z_t)_{0\leq t\leq T}$ such that $\|Z\|_{\mathbf{L^p(W)}} := \Big(\mathbb{E}\Big[\int_0^T |Z_t|^p dt\Big]\Big)^{\frac{1}{p}} < \infty$.
- $\mathbf{L^p}(\tilde{\mu})$, $p \geq 1$, the set of $\mathcal{P} \otimes \mathcal{E}$-measurable maps $U : \Omega \times [0, T] \times E \to \mathbb{R}$ such that $\|U\|_{\mathbf{L^p}(\tilde{\mu})} := \Big(\mathbb{E}[\int_0^T \int_E |U_t(e)|^p \lambda(de)dt\Big]\Big)^{\frac{1}{p}} < \infty$.

- $\mathbf{A^2}$ the closed subset of $\mathcal{S}^{\mathbf{2}}$ consisting of nondecreasing processes $K = (K_t)_{0\leq t\leq T}$ with $K_0 = 0$.

We are given four objects: (i) a terminal function, which is a measurable function $g : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying a growth sublinear condition

$$\sup_{x\in\mathbb{R}^d} \frac{|g(x)|}{1+|x|} \quad < \quad \infty, \tag{1.2.3}$$

(ii) a generator function f, which is a measurable function $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfying a growth sublinear condition

$$\sup_{(x,y,z)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d} \frac{|f(x,y,z)|}{1+|x|+|y|+|z|} \quad < \quad \infty, \tag{1.2.4}$$

and a uniform Lipschitz condition on (y, z), i.e. there exists a constant k_f such that for all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$,

$$|f(x,y,z) - f(x,y',z')| \quad \leq \quad k_f(|y-y'| + |z-z'|), \tag{1.2.5}$$

(iii) a cost function, which is a measurable function $c : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times E \to \mathbb{R}$ satisfying a growth sublinear condition

$$\sup_{(x,y,z,e)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times E} \frac{|c(x,y,z,e)|}{1+|x|+|y|+|z|} \quad < \quad \infty, \tag{1.2.6}$$

and a uniform Lipschitz condition on (y, z), i.e. there exists a constant k_c such that for all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $e \in E$,

$$|c(x,y,z,e) - c(x,y',z',e)| \quad \leq \quad k_c(|y-y'| + |z-z'|), \tag{1.2.7}$$

(iv) a constraint function, which is a measurable map $h : \mathbb{R} \times E \to \mathbb{R}$ s.t for all $e \in E$,

$$u \quad \longmapsto \quad h(u,e) \quad \text{ is nonincreasing,} \tag{1.2.8}$$

satisfying a Lipschitz condition on u i.e. there exists a constant k_h such that for all $u, u' \in \mathbb{R}$, $e \in E$,

$$|h(u,e) - h(u',e)| \quad \leq \quad k_h|u-u'|. \tag{1.2.9}$$

and such that $\int_E |h(0,e)|\lambda(de) < +\infty$.

Let us now introduce our BSDE with constrained jumps: find a quadruple (Y, Z, U, K) $\in \mathcal{S}^{\mathbf{2}} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$ satisfying

$$\begin{aligned} Y_t \quad &= \quad g(X_T) + \int_t^T f(X_s,Y_s,Z_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle \\ &\quad - \int_t^T \int_E (U_s(e) - c(X_{s^-}, Y_{s^-}, Z_s, e))\mu(ds,de), \quad 0 \leq t \leq T, \ a.s. \end{aligned} \tag{1.2.10}$$

with

$$h(U_t(e), e) \;\; \geq \;\; 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(de) \;\; a.e. \tag{1.2.11}$$

and such that for any other quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$ satisfying (1.2.10)-(1.2.11), we have

$$Y_t \;\; \leq \;\; \tilde{Y}_t, \quad 0 \leq t \leq T, \; a.s.$$

We say that Y is the minimal solution to (1.2.10)-(1.2.11). In the formulation of Peng [64], one may sometimes say that Y is the smallest supersolution to (1.2.10)-(1.2.11). We shall also say that (Y, Z, U, K) is a minimal solution to (1.2.10)-(1.2.11), and we discuss later the uniqueness of such quadruple.

Remark 1.2.1 Since we are originally motivated by probabilistic representation of QVI's, we put the BSDE with constrained jumps in a Markovian framework. But all the results of Section 3 about the existence and approximation of a minimal solution hold true in a general non Markovian framework with the following standard modifications : the terminal condition $g(X_T)$ is replaced by a square integrable random variable $\xi \in \mathbf{L^2}(\mathbf{\Omega}, \mathcal{F}_\mathbf{T})$, the generator is a map f from $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}$, satisfying a uniform Lipschitz condition in (y, z), and $f(\cdot, y, z) \in \mathbf{L}^\mathbf{2}_\mathbb{F}(\mathbf{0}, \mathbf{T})$ for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, and the cost coefficient is a map c from $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times E$ into $\mathbb{R}$, satisfying a uniform Lipschitz condition in (y, z), and $c(\cdot, y, z, e) \in \mathbf{L}^\mathbf{2}_\mathbb{F}(\mathbf{0}, \mathbf{T})$ for all $(y, z, e) \in \mathbb{R} \times \mathbb{R}^d \times E$.

Remark 1.2.2 Without the h-constraint condition (1.2.11) on jumps, we have existence and uniqueness of a solution (Y, Z, U, K) with $K = 0$ to (1.2.10), from results on BSDE with jumps in [77] and [5]. Here, under (1.2.11) on jumps, it is not possible in general to have equality in (1.2.10) with $K = 0$, and as usual in the BSDE literature with constraint, we consider a nondecreasing process K to have more freedom. The problem is then to find a minimal solution to this constrained BSDE, and the nondecreasing condition (1.2.8) on h is crucial for stating comparison principles needed in the penalization approach. The primary example of constraint function is $h(u, e) = -u$, i.e. nonpositive jumps constraint, which is actually equivalent to consider minimal solution to BSDE (1.1.13) as showed later.

1.2.2 The case of nonpositive jump constraint

Let us recall the BSDE defined in the introduction: find a triplet $(Y, Z, K) \in \mathcal{S}^\mathbf{2} \times \mathbf{L^2(W)} \times \mathbf{A^2}$ such that

$$\begin{aligned} Y_t \;\; = \;\; & g(X_T) + \int_t^T f(X_s, Y_s, Z_s) ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle \\ & + \int_t^T \int_E c(X_{s^-}, Y_{s^-}, Z_s, e) \mu(ds, de), \quad 0 \leq t \leq T, \; a.s. \end{aligned} \tag{1.2.12}$$

such that for any other triplet $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathcal{S}^2 \times \mathbf{L^2(W)} \times \mathbf{A^2}$ satisfying (1.2.12), it holds that

$$Y_t \quad \leq \quad \tilde{Y}_t, \quad 0 \leq t \leq T, \ a.s.$$

We will call such Y (and, by a slight abuse of notation, (Y, Z, K)) the *minimal solution* to (1.2.12). We claim that this problem is actually equivalent to problem (1.2.10)-(1.2.11) in the case $h(u, e) = -u$, corresponding to nonpositive jump constraint condition:

$$U_t(e) \quad \leq \quad 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(de) \ \ a.e. \tag{1.2.13}$$

Indeed, let (Y, Z, U, K) be any solution of (1.2.10) and (1.2.13). Define a process $\bar{K}$ by $d\bar{K}_t = dK_t - \int_E U_s(e)\mu(dt, de)$, $0 \leq t \leq T$, then $\bar{K}$ is nondecreasing, and the triplet $(Y, Z, \bar{K})$ satisfies (1.2.12). It follows that the minimal solution to (1.2.12) is smaller than the minimal solution to (1.2.10) and (1.2.13). We shall see in the next section, by using comparison principles and penalization approach, that equality holds, i.e.

$$\text{minimal solution } Y \text{ to } (1.2.12) \quad = \quad \text{minimal solution } Y \text{ to } (1.2.10), (1.2.13).$$

We shall illustrate this result by considering a special case : when the functions f and c do not depend on y, z (i.e., the impulse control case). In this case, one can obtain directly the existence of a minimal solution to (1.2.10)-(1.2.13) and (1.2.12) by duality methods involving the following set of probability measures. Let $\mathcal{V}$ be the set of $\mathcal{P} \otimes \mathcal{E}$-measurable essentially bounded processes valued in $(0, \infty)$, and given $\nu \in \mathcal{V}$, consider the probability measure $\mathbb{P}^\nu$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with Radon-Nikodym density :

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}} \quad = \quad \mathcal{E}_T\Big(\int_0^. \int_E (\nu_t(e) - 1)\tilde{\mu}(dt, de)\Big), \tag{1.2.14}$$

where $\mathcal{E}_t(.)$ is the Doléans-Dade exponential. Notice that the Brownian motion W remains a Brownian motion under $\mathbb{P}^\nu$, which can then be interpreted as an equivalent martingale measure for the "asset" price process W. The effect of the probability measure $\mathbb{P}^\nu$, by Girsanov's theorem, is to change the compensator $\lambda(de)dt$ of μ under $\mathbb{P}$ to $\nu_t(e)\lambda(de)dt$ under $\mathbb{P}^\nu$.

In order to ensure that the problem is well-defined, we need to assume :

(H1) There exists a triple $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathcal{S}^2 \times \mathbf{L^2(W)} \times \mathbf{A^2}$ satisfying (1.2.12).

This assumption is standard and natural in the literature on BSDE with constraints, and means equivalently here (when f and c do not depend on y, z) that one can find some constant $\tilde{y} \in \mathbb{R}$, and $\tilde{Z} \in \mathbf{L^2(W)}$ such that

$$\tilde{y} + \int_0^T \langle \tilde{Z}_s, dW_s \rangle \quad \geq \quad g(X_T) + \int_0^T f(X_s)ds + \int_0^T \int_E c(X_{s^-}, e)\mu(ds, de) \quad a.s.$$

This equivalency can be proved by same arguments as in [24]. Notice that Assumption **(H1)** may be not satisfied as shown in Remark 1.3.1, in which case the problem (1.2.12) is ill-posed.

Theorem 1.2.1 *Suppose that f and c do not depend on y, z, and* **(H1)** *holds. Then, there exists a unique minimal solution $(Y, Z, K, U) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$, with K predictable, to* (1.2.10)-(1.2.13). *Moreover, $(Y, Z, \bar{K})$ is the unique minimal solution to* (1.2.12) *with $\bar{K}_t = K_t - \int_0^t \int_E U_s(e)\mu(ds, de)$, and Y has the explicit functional representation :*

$$Y_t \;=\; \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^{\nu}\Big[g(X_T) + \int_t^T f(X_s)ds + \int_t^T \int_E c(X_{s^-}, e)\mu(ds, de)\Big|\mathcal{F}_t\Big],$$

for all $t \in [0, T]$.

Proof. First, observe that for any $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$ (resp. $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L^2(W)} \times \mathbf{A^2}$) satisfying (1.2.10)-(1.2.13) (resp. (1.2.12)), the process

$$\tilde{Q}_t \;:=\; \tilde{Y}_t + \int_0^t f(X_s)ds + \int_0^t \int_E c(X_{s^-}, e)\mu(ds, de), \quad 0 \le t \le T,$$

is a $\mathbb{P}^{\nu}$-supermartingale, for all $\nu \in \mathcal{V}$, where the probability measure $\mathbb{P}^{\nu}$ was defined in (1.2.14). Indeed, from (1.2.10)-(1.2.13) (resp. (1.2.12)), we have

$$\tilde{Q}_t \;=\; \tilde{Q}_0 + \int_0^t \langle \tilde{Z}_s, dW_s\rangle - \bar{K}_t, \text{ with } \bar{K}_t = \tilde{K}_t - \int_0^t U_s(e)\mu(ds, de),$$

$$(\text{ resp. } \tilde{Q}_t \;=\; \tilde{Q}_0 + \int_0^t \langle \tilde{Z}_s, dW_s\rangle - \tilde{K}_t), \quad 0 \le t \le T.$$

Now, by Girsanov's theorem, W remains a Brownian motion under $\mathbb{P}^{\nu}$, while from the boundedness of $\nu \in \mathcal{V}$, the density $d\mathbb{P}^{\nu}/d\mathbb{P}$ lies in $L^2(\mathbb{P})$. Hence, from Cauchy-Schwarz inequality, the condition $\tilde{Z} \in \mathbf{L^2(W)}$, and Burkholder-Davis-Gundy inequality, we get the $\mathbb{P}^{\nu}$-martingale property of the stochastic integral $\int \langle \tilde{Z}, dW\rangle$, and so the $\mathbb{P}^{\nu}$-supermartingale property of $\tilde{Q}$ since $\bar{K}$ (resp. $\tilde{K}$) is nondecreasing. This implies

$$\tilde{Y}_t \;\ge\; \mathbb{E}^{\nu}\Big[\tilde{Y}_T + \int_t^T f(X_s)ds + \int_t^T \int_E c(X_{s^-}, e)\mu(ds, de)\Big|\mathcal{F}_t\Big],$$

and thereby, from the arbitrariness of $\mathbb{P}^{\nu}$, $\nu \in \mathcal{V}$, and since $\tilde{Y}_T = g(X_T)$,

$$\begin{aligned} Y_t \;&:=\; \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^{\nu}\Big[g(X_T) + \int_t^T f(X_s)ds + \int_t^T \int_E c(X_{s^-}, e)\mu(ds, de)\Big|\mathcal{F}_t\Big] \qquad (1.2.15) \\ &\le\; \tilde{Y}_t. \end{aligned}$$

To show the converse, let us consider the process Y defined in (1.2.15). By standard arguments as in [30], the process Y can be considered in its càd-làg modification, and we also notice that $Y \in \mathcal{S}^{\mathbf{2}}$. Indeed, by observing that the choice of $\nu = 1$ corresponds to the probability $\mathbb{P}^{\nu} = \mathbb{P}$, we have $\hat{Y} \le Y \le \tilde{Y}$, where $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L^2(W)} \times \mathbf{A^2}$ is a solution to (1.2.12), and

$$\hat{Y}_t \;=\; \mathbb{E}\Big[g(X_T) + \int_t^T f(X_s)ds + \int_t^T \int_E c(X_{s^-}, e)\mu(ds, de)\Big|\mathcal{F}_t\Big].$$

Thus, since $\hat{Y}$ lies in $\mathcal{S}^{\mathbf{2}}$ from the linear growth conditions on g, f, and c, and the estimate (1.2.2), we deduce that $Y \in \mathcal{S}^{\mathbf{2}}$. Now, by similar dynamic programming arguments as in [30], we see that the process

$$Q_t = Y_t + \int_0^t f(X_s)ds + \int_0^t \int_E c(X_{s^-}, e)\mu(ds, de), \quad 0 \leq t \leq T, \tag{1.2.16}$$

lies in $\mathcal{S}^{\mathbf{2}}$, and is a $\mathbb{P}^\nu$-supermartingale, for all $\nu \in \mathcal{V}$. Then, from the Doob-Meyer decomposition of Q under each $\mathbb{P}^\nu$, $\nu \in \mathcal{V}$, we obtain :

$$Q_t = Y_0 + M^\nu - K^\nu, \tag{1.2.17}$$

where M^ν is a $\mathbb{P}^\nu$-martingale, $M_0^\nu = 0$, and K^ν is a $\mathbb{P}^\nu$ nondecreasing predictable càd-làg process with $K_0^\nu = 0$. Recalling that W is a $\mathbb{P}^\nu$-Brownian motion, and since $\tilde{\mu}^\nu(ds, de)$ $:= \mu(ds, de) - \nu_s(e)\lambda(de)ds$ is the compensated measure of μ under $\mathbb{P}^\nu$, the martingale representation theorem for each M^ν, $\nu \in \mathcal{V}$ gives the existence of predictable processes Z^ν and U^ν such that

$$\begin{aligned} Q_t &= Y_0 + \int_0^t \langle Z_s^\nu, dW_s\rangle \\ &\quad + \int_0^t \int_E U_s^\nu(e)\tilde{\mu}^\nu(ds, de) - K_t^\nu, \quad 0 \leq t \leq T. \end{aligned} \tag{1.2.18}$$

By comparing the decomposition (1.2.18) under $\mathbb{P}^\nu$ and $\mathbb{P}$ corresponding to $\nu = 1$, and identifying the martingale parts and the predictable finite variation parts, we obtain that $Z^\nu = Z^1 =: Z$, $U^\nu = U^1 =: U$ for all $\nu \in \mathcal{V}$, and

$$K_t^\nu = K_t^1 - \int_0^t \int_E U_s(e)(\nu_s(e) - 1)\lambda(de)ds, \quad 0 \leq t \leq T. \tag{1.2.19}$$

Now, by writing the relation (1.2.18) with $\nu = \varepsilon > 0$, substituting the definition of Q in (1.2.16), and since $Y_T = g(X_T)$, we obtain :

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s)ds - \int_t^T \langle Z_s, dW_s\rangle \\ &\quad - \int_t^T \int_E (U_s(e) - c(X_{s^-}, e))\mu(ds, de) \\ &\quad + \int_t^T \int_E U_s(e)\varepsilon\lambda(de)ds + K_T^\varepsilon - K_t^\varepsilon, \quad 0 \leq t \leq T. \end{aligned} \tag{1.2.20}$$

From (1.2.19), the process K^ε has a limit as ε goes to zero, which is equal to $K^0 = K^1 + \int_0^{\cdot} \int_E U_s(e)\lambda(de)ds$, and inherits from K^ε, the nondecreasing path and predictability properties. Moreover, since $Q \in \mathcal{S}^{\mathbf{2}}$, in the decomposition (1.2.17) of Q under $\mathbb{P} = \mathbb{P}^\nu$ for $\nu = 1$, the process M^1 lies in $\mathcal{S}^{\mathbf{2}}$ and $K^1 \in \mathbf{A^2}$. This implies that $Z \in \mathbf{L^2(W)}$, $U \in \mathbf{L^2}(\tilde{\mu})$, and also that $K^0 \in \mathbf{A^2}$. By sending ε to zero into (1.2.20), we obtain that $(Y, Z, U, K^0) \in$

$\mathcal{S}^2 \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$ is a solution to (1.2.10). Let us finally check that U satisfies the constraint :

$$U_t(e) \;\; \leq \;\; 0, \quad d\mathbb{P} \otimes dt \otimes \lambda(de). \tag{1.2.21}$$

We argue by contradiction by assuming that the set $F = \{(\omega, t, e) \in \Omega \times [0, T] \times E \; : U_t(e) > 0\}$ has a strictly positive measure for $d\mathbb{P} \times dt \times \lambda(de)$. For any $k > 0$, consider the process $\nu_k = 1_{F^c} + (k+1)1_F$, which lies in $\mathcal{V}$. From (1.2.19), we have

$$\mathbb{E}[K_T^{\nu_k}] \;\; = \;\; \mathbb{E}[K_T^1] - k\mathbb{E}\Big[\int_0^T \int_E 1_F U_t(e)\lambda(de)dt\Big] \; < \; 0,$$

for k large enough. This contradicts the fact that $K_T^{\nu_k} \geq 0$, and so (1.2.21) is satisfied. Therefore (Y, Z, U, K^0) is a solution to (1.2.10)-(1.2.13), and it is a minimal solution from (1.2.15). Y is unique by definition. The uniqueness of Z follows by identifying the Brownian parts and the finite variation parts, and the uniqueness of (U, K^0) is obtained by identifying the predictable parts by recalling that the jumps of μ are inaccessible. By denoting $\bar{K}^0 = K^0 - \int_0^t \int_E U_s(e)\mu(ds, de)$, which lies in $\mathbf{A^2}$, we see that $(Y, Z, \bar{K}^0)$ is a solution to (1.2.12), and it is minimal by (1.2.15). Uniqueness follows by identifying the Brownian parts and the finite variation parts. □

Remark 1.2.3 In Section 1.4, we shall relate rigorously the constrained BSDEs (1.2.10)-(1.2.11) to QVIs. In particular, the minimal solution Y_t to (1.2.10)-(1.2.13) or (1.2.12) is $Y_t = v(t, X_t)$ where v is the value function of the impulse control problem (1.1.4). Together with the functional representation of Y in Theorem 1.2.1, we then have the following relation at time $t = 0$:

$$v(0, X_0) \;\; = \;\; \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \Big[g(X_T) + \int_0^T f(X_s)ds + \int_0^T \int_E c(X_{s^-}, e)\mu(ds, de)\Big]. \tag{1.2.22}$$

We then recover a recent result obtained by Bouchard [10], who related impulse controls to stochastic target problems in the case of a finite set E. We may also interpret this result as follows. Recall that the effect of the probability measure $\mathbb{P}^\nu$ is to change the compensator $\lambda(de)dt$ of μ under $\mathbb{P}$ to $\nu_t(e)\lambda(de)dt$ under $\mathbb{P}^\nu$. Hence, by taking the supremum over all $\mathbb{P}^\nu$, we formally expect to retrieve in distribution law all the dynamics of the controlled process in (1.1.5) when varying the impulse controls α, which is confirmed by the equality (1.2.22).

Finally, we mention that the above duality and martingale methods may be extended when the generator function f is linear in z by using Girsanov's transformation. Our main purpose is now to study the general case of h-constraints on jumps, and nonlinear functions f and c depending on y, z.

1.3 Existence and approximation by penalization

In this section, we prove the existence of a minimal solution to (1.2.10)-(1.2.11), based on approximation via penalization. For each $n \in \mathbb{N}$, we introduce the penalized BSDE with jumps

$$
\begin{aligned}
Y_t^n &= g(X_T) + \int_t^T f(X_s, Y_s^n, Z_s^n) ds \\
&\quad + n \int_t^T \int_E h^-(U_s^n(e), e) \lambda(de) ds - \int_t^T \langle Z_s^n, dW_s \rangle \\
&\quad - \int_t^T \int_E (U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)) \mu(ds, de), \quad 0 \le t \le T,
\end{aligned} \tag{1.3.1}
$$

where $h^-(u, e) = \max(-h(u, e), 0)$ is the negative part of the function h. Under the Lipschitz and growth conditions on the coefficients f, c and h, we know from the theory of BSDEs with jumps, see [77] and [5], that there exists a unique solution $(Y^n, Z^n, U^n) \in \mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ to (1.3.1). We define for each $n \in \mathbb{N}$,

$$
K_t^n = n \int_0^t \int_E h^-(U_s^n(e), e) \lambda(de) ds, \quad 0 \le t \le T,
$$

which is a nondecreasing process in $\mathbf{A}^2$. The rest of this section is devoted to the convergence of the sequence $(Y^n, Z^n, U^n, K^n)_n$ to the minimal solution we are interested in.

1.3.1 Comparison results

We first state that the sequence $(Y^n)_n$ is nondecreasing. This follows from a comparison theorem for BSDEs with jumps whose generator is of the form $\tilde{f}(x, y, z, u) = f(x, y, z) + \int_E \tilde{h}(u(e), e) \lambda(de)$ for some nondecreasing function $\tilde{h}$, which covers our situation from the nonincreasing condition on the constraint function h.

Lemma 1.3.1 *The sequence $(Y^n)_n$ is nondecreasing, i.e. for all $n \in \mathbb{N}$, $Y_t^n \le Y_t^{n+1}$, $0 \le t \le T$, a.s.*

Proof. Define the sequence $(V^n)_n$ of $\mathcal{P} \otimes \mathcal{E}$-measurable processes by

$$
\begin{aligned}
V_t^n(e) &= U_t^n(e) - c(X_{t^-}, Y_{t^-}^n, Z_t^n, e), \quad (t, e) \in (0, T] \times E \quad \text{and} \\
V_0^n(e) &= U_0^n(e) - c(X_0, Y_0^n, Z_0^n, e), \quad e \in E,
\end{aligned}
$$

From (1.3.1) and recalling that X and Y are càd-làg, we see that (Y^n, Z^n, V^n) is the unique solution in $\mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ of the BSDE with jumps :

$$
\begin{aligned}
Y_t^n &= g(X_T) + \int_t^T F_n(X_s, Y_s^n, Z_s^n, V_s^n) ds \\
&\quad - \int_t^T \langle Z_s^n, dW_s \rangle - \int_t^T \int_E V_s^n(e) \tilde{\mu}(ds, de),
\end{aligned}
$$

with $F_n(x, y, z, v) = f(x, y, z) + \int_E (n h^-(v(e) + c(x, y, z, e), e) - v(e)) \lambda(de)$. Since h^- is nondecreasing, we have

$$
F_n(t, x, y, z, v) - F_n(t, x, y, z, v') =
$$

$$\int_E \big\{(v'(e) - v(e)) + n[h^-(v(e) + c(x,y,z,e),e) \\ - h^-(v'(e) + c(x,y,z,e),e)]\big\}\,\lambda(de) \quad \leq \\ \int_E \big\{(-1 + \mathbf{1}_{\{v(e)\geq v'(e)\}} n k_h)(v(e) - v'(e))\big\}\,\lambda(de).$$

Moreover, since $F_{n+1} \geq F_n$, we can apply the comparison theorem 2.5 of [73], and obtain that $Y_t^n \leq Y_t^{n+1}$, $0 \leq t \leq T$, a.s. □

The next result shows that the sequence $(Y^n)_n$ is upper-bounded by any solution to the constrained BSDE. Arguments in the proof involve suitable change of probability measures $\mathbb{P}^\nu$, $\nu \in \mathcal{V}$, introduced in (1.2.14).

Lemma 1.3.2 *For any quadruple $(\tilde Y, \tilde Z, \tilde U, \tilde K) \in \mathcal{S}^2 \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde\mu) \times \mathbf{A^2}$ satisfying* (1.2.10)-(1.2.11)*, and for all $n \in \mathbb{N}$, we have*

$$Y_t^n \quad \leq \quad \tilde Y_t, \quad 0 \leq t \leq T, \ a.s. \tag{1.3.2}$$

Moreover, in the case : $h(u,e) = -u$, the inequality (1.3.2) *also holds for any triple $(\tilde Y, \tilde Z, \tilde K)$ $\in \mathcal{S}^2 \times \mathbf{L^2(W)} \times \mathbf{A^2}$ satisfying* (1.2.12)*.*

Proof. We state the proof for quadruple $(\tilde Y, \tilde Z, \tilde U, \tilde K)$ satisfying (1.2.10)-(1.2.11). Same arguments are used in the case : $h(u,e) = -u$ and $(\tilde Y, \tilde Z, \tilde K) \in \mathcal{S}^2 \times \mathbf{L^2(W)} \times \mathbf{A^2}$ satisfying (1.2.12).

Denote $\bar Y = \tilde Y - Y^n$, $\bar Z = \tilde Z - Z^n$, $\bar f = f(X, \tilde Y, \tilde Z) - f(X, Y^n, Z^n)$ and $\bar c = c(X_{\cdot^-}, \tilde Y_{\cdot^-}, \tilde Z, e) - c(X_{\cdot^-}, Y^n_{\cdot^-}, Z^n, e)$. Fix some $\nu \in \mathcal{V}$ (to be chosen later). We then have :

$$\begin{aligned} \bar Y_t \ = \ & \int_t^T \bar f_s ds + \int_t^T\int_E \bar c_s \mu(ds,de) - \int_t^T \langle \bar Z_s, dW_s\rangle \\ & - \int_t^T\int_E \Big\{\tilde U_s(e) - U_s^n(e)\Big\}\,\tilde\mu^\nu(ds,de) - \int_t^T\int_E \Big\{\tilde U_s(e) - U_s^n(e)\Big\}\,\nu_s(e)\lambda(de)ds \\ & - n\int_t^T\int_E h^-(U_s^n(e),e)\lambda(de)ds + \tilde K_T - \tilde K_t, \end{aligned}$$

$$\begin{aligned} \bar Y_t \ = \ & \int_t^T \bar f_s ds + \int_t^T\int_E \bar c_s \mu(ds,de) - \int_t^T \langle \bar Z_s, dW_s\rangle \\ & - \int_t^T\int_E \Big\{\tilde U_s(e) - U_s^n(e)\Big\}\,\tilde\mu^\nu(ds,de) - \int_t^T\int_E \Big\{\tilde U_s(e) - U_s^n(e)\Big\}\,\nu_s(e)\lambda(de)ds \\ & - n\int_t^T\int_E h^-(U_s^n(e),e)\lambda(de)ds + \tilde K_T - \tilde K_t, \end{aligned}$$

where $\tilde\mu^\nu(dt,de) = \mu(dt,de) - \nu_t(e)\lambda(de)dt$ denotes the compensated measure of μ under $\mathbb{P}^\nu$. Let us then define the following adapted processes:

$$a_t \quad = \quad \frac{f(X_t, \tilde Y_t, \tilde Z_t) - f(X_t, Y_t^n, \tilde Z_t)}{\bar Y_t}\mathbf{1}_{\{\bar Y_t \neq 0\}},$$

and b the $\mathbb{R}^d$-valued process defined by its i-th components, $i = 1, \ldots, d$:

$$b_t^i = \frac{f(X_t, Y_t^n, Z_t^{(i-1)}) - f(X_t, Y_t^n, Z_t^{(i)})}{V_t^i} \mathbf{1}_{\{V_t^i \neq 0\}},$$

where $Z_t^{(i)}$ is the $\mathbb{R}^d$-valued random vector whose i first components are those of $\tilde{Z}$ and whose $(d-i)$ lasts are those of Z^n, and V_t^i is the i-th component of $Z_t^{(i-1)} - Z_t^{(i)}$. Let us also define the $\mathcal{P} \otimes \mathcal{E}$-measurable processes δ in $\mathbb{R}$ and ℓ in $\mathbb{R}^d$ by:

$$\delta_t(e) = \frac{c(X_{t^-}, \tilde{Y}_{t^-}, \tilde{Z}_t, e) - c(X_{t^-}, Y_{t^-}^n, \tilde{Z}_t, e)}{\bar{Y}_r} \mathbf{1}_{\{\bar{Y}_{t^-} \neq 0\}},$$

and

$$\ell_r^i(e) = \frac{c(X_{t^-}, Y_{t^-}^n, Z_t^{(i-1)}, e) - c(X_{t^-}, Y_{t^-}^n, Z_t^{(i)}, e)}{V_t^i} \mathbf{1}_{\{V_t^i \neq 0\}}.$$

Notice that the processes a, b, δ and ℓ are bounded by the Lipschitz conditions on f and c. Define also $\alpha_t^\nu = a_t + \int_E \delta_t(e)\nu_t(e)\lambda(de)$, $\beta_t^\nu = b_t + \int_E \ell_t(e)\nu_t(e)\lambda(de)$, which are bounded processes since a, b, δ, ℓ are bounded and λ is a finite measure on E, and denote $V_t^n(e) = \tilde{U}_t(e) - U_t^n(e) - \delta_t(e)\bar{Y}_t - \ell_t(e) \cdot \bar{Z}_t$. With these notations, and recalling that $h^-(\tilde{U}_s(e), e) = 0$ from the constraint condition (1.2.11), we rewrite the BSDE for $\bar{Y}$ as:

$$\begin{aligned}\bar{Y}_t =& \int_t^T (\alpha_s^\nu \bar{Y}_s + \beta_s^\nu . \bar{Z}_s) ds - \int_t^T \langle \bar{Z}_s, dW_s \rangle - \int_t^T \int_E V_s^n(e) \tilde{\mu}^\nu(ds, de) + \tilde{K}_T - \tilde{K}_t \\ &+ \int_t^T \int_E \Big\{ n[h^-(\tilde{U}_s(e), e) - h^-(U_s^n(e), e)] - \nu_s(e)[\tilde{U}_s(e) - U_s^n(e)] \Big\} \lambda(de) ds.\end{aligned}$$

Consider now the positive process Γ^ν solution to the s.d.e.:

$$d\Gamma_t^\nu = \Gamma_t^\nu(\alpha_t^\nu dt + \langle \beta_t^\nu, dW_t \rangle), \quad \Gamma_0^\nu = 1, \tag{1.3.3}$$

and notice that Γ^ν lies in $\mathcal{S}^2$ from the boundeness condition on α^ν and β^ν. By Itô's formula, we have

$$\begin{aligned}d\Gamma_t^\nu \bar{Y}_t =& -\Gamma_t^\nu \int_E \Big\{ n[h^-(\tilde{U}_t(e), e) - h^-(U_t^n(e), e)] - \nu_t(e)[\tilde{U}_t(e) - U_t^n(e)] \Big\} \lambda(de) ds \\ & -\Gamma_t^\nu d\tilde{K}_t + \Gamma_t^\nu \langle \bar{Z}_t, dW_t \rangle + \Gamma_t^\nu \bar{Y}_{t^-} \langle \beta_t, dW_t \rangle + \Gamma_t^\nu \int_E V_t^n(e) \tilde{\mu}^\nu(dt, de),\end{aligned}$$

which shows that the process

$$\Gamma_t^\nu \bar{Y}_t + \int_0^t \Gamma_s^\nu \int_E \Big\{ n[h^-(\tilde{U}_s(e), e) - h^-(U_s^n(e), e)] - \nu_s(e)[\tilde{U}_s(e) - U_s^n(e)] \Big\} \lambda(de) ds$$

is a $\mathbb{P}^\nu$-supermartingale and so

$$\begin{aligned}\Gamma_t^\nu \bar{Y}_t \geq& \; \mathbb{E}^\nu \Big[\int_t^T \Gamma_s^\nu \int_E \Big\{ n[h^-(\tilde{U}_s(e), e) - h^-(U_s^n(e), e)] \\ & -\nu_s^\varepsilon(e)[\tilde{U}_s(e) - U_s^n(e)] \Big\} \lambda(de) ds \Big| \mathcal{F}_t \Big].\end{aligned}$$

Now, from the Lipschitz condition on h, we see that the process ν^ε defined by

$$\nu_t^\varepsilon(e) = \begin{cases} \frac{n[h^-(\tilde{U}_t(e),e)-h^-(U_t^n(e),e)]}{\tilde{U}_t(e)-U_t^n(e)} & \text{if } U_t^n(e) > \tilde{U}_t(e) \text{ and } h^-(U_t^n(e),e) > 0 \\ \varepsilon & \text{else} \end{cases}$$

is bounded and so lies in $\mathcal{V}$, and therefore by taking $\nu = \nu^\varepsilon$, we obtain :

$$\begin{aligned} \Gamma_t^{\nu^\varepsilon} \bar{Y}_t &\geq -\varepsilon \mathbb{E}^{\nu^\varepsilon}\Big[\int_t^T \Gamma_s^{\nu^\varepsilon} \int_E [\tilde{U}_s(e) - U_s^n(e)] \\ &\qquad \mathbf{1}_{\{\tilde{U}_s(e) \geq U_s^n(e)\} \cup \{h^-(U_s^n(e),e)=0\}} \lambda(de) ds \Big| \mathcal{F}_t\Big] \\ &=: -\varepsilon R_t^\varepsilon, \quad 0 \leq t \leq T. \end{aligned} \tag{1.3.4}$$

From the conditional Cauchy-Schwarz inequality, and Bayes formula, we have for all $t \in [0,T]$, $\varepsilon > 0$,

$$\begin{aligned} |R_t^\varepsilon| &\leq \sqrt{\mathbb{E}\left[\frac{Z_T^{\nu^\varepsilon}}{Z_t^{\nu^\varepsilon}} \int_t^T |\Gamma_s^{\nu^\varepsilon}|^2 ds \Big| \mathcal{F}_t\right]} \\ &\quad . \sqrt{\mathbb{E}\left[\frac{Z_T^{\nu^\varepsilon}}{Z_t^{\nu^\varepsilon}} \int_t^T \Big(\int_E [\tilde{U}_s(e) - U_s^n(e)] \mathbf{1}_{\{\tilde{U}_s(e) \geq U_s^n(e)\} \cup \{h^-(U_s^n(e),e)=0\}} \lambda(de)\Big)^2 ds \Big| \mathcal{F}_t\right]} \\ &=: R_t^{1,\varepsilon} . R_t^{2,\varepsilon}. \end{aligned}$$

By definition of ν^ε, we have for $\varepsilon \leq n k_h$:

$$\frac{Z_T^{\nu^\varepsilon}}{Z_t^{\nu^\varepsilon}} \leq \frac{Z_T^n}{Z_t^n} \exp\Big(\int_t^T \int_E n k_h \lambda(de) ds\Big),$$

where Z^n is the solution to $dZ_t^n = Z_{t-}^n \int_E (n k_h - 1)\, \tilde{\mu}(dt, de)$, $Z_0^n = 1$. It follows that for all $t \in [0,T]$, $(R_t^{2,\varepsilon})_\varepsilon$ is uniformly bounded for ε in a neighborhood of 0^+. Similarly, using also the boundedness of the coefficients α^{ν^ε} and β^{ν^ε} in the dynamics (1.3.3) of $\Gamma^{\nu,\varepsilon}$, we deduce that $(R_t^{1,\varepsilon})_\varepsilon$ and thus $(R_t^\varepsilon)_\varepsilon$ is uniformly bounded for ε in a neighborhood of 0^+. Finally, since $\lim_{\varepsilon \to 0} \Gamma_t^{\nu^\varepsilon} = \Gamma_t^{\nu^0} > 0$, by sending ε to zero into (1.3.4), we conclude that $\bar{Y}_t \geq 0$. □

1.3.2 Convergence of the penalized BSDEs

We impose the following analogue of Assumption **(H1)**.

(H2) There exists a quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U}) \in \mathcal{S}^\mathbf{2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$ satisfying (1.2.10)-(1.2.11).

Assumption **(H2)** ensures that the problem (1.2.10)-(1.2.11) is well-posed. As indicated in paragraph 1.2.2, Assumption **(H2)** in the case $h(u,e) = -u$, implies Assumption **(H1)**. Since **(H1)** is obviously stronger than **(H2)**, these two Assumptions are equivalent in the case $h(u,e) = -u$. We provide in Section 1.5 some discussion and sufficient conditions under which **(H2)** holds.

Remark 1.3.1 The following example shows that conditions **(H1)** and **(H2)** may be not satisfied : consider the BSDEs

$$Y_t = -\int_t^T \langle Z_s, dW_s\rangle + \int_t^T \int_E c\mu(ds,de) + K_T - K_t, \tag{1.3.5}$$

and

$$\begin{cases} Y_t = -\int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_E [U_s(e) - c]\mu(ds,de) + K_T - K_t \\ -U_s(e) \geq 0 \end{cases} \tag{1.3.6}$$

where c is a strictly positive constant, $c > 0$. Then, there does not exist any solution to (1.3.5) or (1.3.6) with component $Y \in \mathcal{S}^{\mathbf{2}}$. On the contrary, we would have

$$Y_0 \geq -\int_0^T \langle Z_s, dW_s\rangle + c\mu([0,T]\times E), \qquad \text{a.s.}$$

which implies that for all $n \in \mathbb{N}^*$, $\nu \equiv n \in \mathcal{V}$,

$$Y_0 \geq \mathbb{E}^\nu\left[-\int_0^T \langle Z_s, dW_s\rangle + c\mu([0,T]\times E)\right] = cn\lambda(E)T.$$

By sending n to infinity, we get the contradiction : $\|Y\|_{\mathcal{S}^2} = \infty$.

We now establish a priori estimates, uniform on n, on the sequence $(Y^n, Z^n, U^n, K^n)_n$.

Lemma 1.3.3 *Under* **(H2)** *(or* **(H1)** *in the case : $h(u,e) = -u$), there exists some constant C such that*

$$\|Y^n\|_{\mathcal{S}^{\mathbf{2}}} + \|Z^n\|_{\mathbf{L^2(W)}} + \|U^n\|_{\mathbf{L^2(\tilde{\mu})}} + \|K^n\|_{\mathcal{S}^{\mathbf{2}}} \leq C, \qquad \forall n \in \mathbb{N}. \tag{1.3.7}$$

Proof. In what follows we shall denote $C > 0$ to be a generic constant depending only on T, the coefficients f, c, the process X, and the bound for $\tilde{Y}$ in **(H1)** or **(H2)**, and which may vary from line to line.

Applying Itô's formula to $|Y_t^n|^2$, and observing that K^n is continuous and $\Delta Y_t^n = \int_E \{U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)\}\mu(\{t\}, de)$, we have

$$\begin{aligned}\mathbb{E}|g(X_T)|^2 = {} & \mathbb{E}|Y_t^n|^2 - 2\mathbb{E}\int_t^T Y_s^n f(X_s, Y_s^n, Z_s^n)ds \\ & -2\mathbb{E}\int_t^T Y_s^n dK_s^n + \mathbb{E}\int_t^T |Z_s^n|^2 ds \\ & +\mathbb{E}\int_t^T\int_E \left\{|Y_{s^-}^n + U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)|^2 - |Y_{s-}^n|^2\right\}\lambda(de)ds\end{aligned}$$

From the linear growth condition on f and the inequality $Y_t^n \leq \tilde{Y}_t$ by Lemma 1.3.2 under **(H2)** (and also under **(H1)** in the case $h(u,e) = -u$), and using the inequality $2ab \leq \frac{1}{\alpha}a^2$

$+ \alpha b^2$ for any constant $\alpha > 0$, we have:

$$
\begin{aligned}
& \mathbb{E}|Y_t^n|^2 + \mathbb{E}\int_t^T |Z_s^n|^2 ds + \mathbb{E}\int_t^T\int_E |U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)|^2 \lambda(de)ds \\
\leq\ & \mathbb{E}|g(X_T)|^2 + 2C\mathbb{E}\int_t^T |Y_s^n|\,(1 + |X_s| + |Y_s^n| + |Z_s^n|)\,ds \\
& -2\mathbb{E}\int_t^T\int_E Y_{s-}^n(U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e))\lambda(de)ds \\
& +\frac{1}{\alpha}\mathbb{E}\Big[\sup_{t\in[0,T]} |\tilde{Y}_t|^2\Big] + \alpha\mathbb{E}|K_T^n - K_t^n|^2.
\end{aligned}
$$

Using again the inequality $2ab \leq \frac{1}{\eta}a^2 + \eta b^2$ for $\eta > 0$, yields

$$
\begin{aligned}
& \mathbb{E}|Y_t^n|^2 + \mathbb{E}\int_t^T |Z_s^n|^2 ds \\
& +\frac{1-\eta}{2}\mathbb{E}\int_t^T\int_E |U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)|^2 \lambda(de)ds \\
\leq\ & \mathbb{E}|g(X_T)|^2 + 2C\mathbb{E}\int_t^T |Y_s^n|\,(1 + |X_s| + |Y_s^n| + |Z_s^n|)\,ds \\
& + \frac{\lambda(E)}{\eta}\mathbb{E}\int_t^T |Y_s^n|^2 ds + \frac{1}{\alpha}\mathbb{E}\Big[\sup_{t\in[0,T]} |\tilde{Y}_t|^2\Big] + \alpha\mathbb{E}|K_T^n - K_t^n|^2 \\
\leq\ & C\Big(1 + \mathbb{E}\int_t^T |Y_s^n|^2 ds\Big) + \frac{1}{2}\mathbb{E}\int_t^T |Z_s^n|^2 ds \\
& +\alpha\mathbb{E}|K_T^n - K_t^n|^2 + \frac{\lambda(E)}{\eta}\mathbb{E}\int_t^T |Y_s^n|^2 ds.
\end{aligned}
$$

Then, by using the inequality $(a-b)^2 \geq a^2/2 - b^2$, we get

$$
\begin{aligned}
& \mathbb{E}|Y_t^n|^2 + \frac{1}{2}\mathbb{E}\int_t^T |Z_s^n|^2 ds + \frac{1-\eta}{4}\mathbb{E}\int_t^T\int_E |U_s^n(e)|^2 \lambda(de)ds \\
\leq\ & \frac{1-\eta}{2}\mathbb{E}\int_t^T\int_E |c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)|^2 \lambda(de)ds \\
& + C\Big(1 + \mathbb{E}\int_t^T |Y_s^n|^2 ds\Big) + \alpha\mathbb{E}|K_T^n - K_t^n|^2 \\
\leq\ & C\Big(1 + \mathbb{E}\int_t^T |Y_s^n|^2 ds\Big) + C(1-\eta)\mathbb{E}\int_t^T |Z_s^n|^2 ds + \alpha\mathbb{E}|K_T^n - K_t^n|^2, \qquad (1.3.8)
\end{aligned}
$$

from the linear growth condition on c. Now, from the relation

$$
\begin{aligned}
K_T^n - K_t^n \ =\ & Y_t^n - g(X_T) - \int_t^T f(X_s, Y_s^n, Z_s^n)ds \\
& + \int_t^T \int_E (U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n))\mu(ds, de) + \int_t^T \langle Z_s^n, dW_s\rangle,
\end{aligned}
$$

and the linear growth condition on f, c, there exists some positive constant C_1 s.t.

$$
\mathbb{E}|K_T^n - K_t^n|^2 \qquad (1.3.9)
$$

$$\leq \quad C_1 \left(1 + \mathbb{E}|Y_t^n|^2 + \mathbb{E} \int_t^T (|Y_s^n|^2 + |Z_s^n|^2) ds + \mathbb{E} \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds \right).$$

Hence, by choosing $\eta > 0$ s.t. $\left(\frac{1}{2} - C(1-\eta)\right) \wedge \left(\frac{1-\eta}{2}\right) > 0$ and $\alpha > 0$ s.t. $C_1 \alpha < \left(\frac{1}{2} - C(1-\eta)\right) \wedge \left(\frac{1-\eta}{2}\right)$, and plugging into (1.3.8), we get

$$\mathbb{E}|Y_t^n|^2 + \mathbb{E} \int_t^T |Z_s^n|^2 ds + \mathbb{E} \int_t^T \int_E |U_s^n(e)|^2 \lambda(de) ds \quad \leq \quad C \left(1 + \mathbb{E} \int_t^T |Y_s^n|^2 ds \right).$$

By applying Gronwall's lemma to $t \mapsto \mathbb{E}|Y_t^n|^2$ and (1.3.9), we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E}|Y_t^n|^2 + \mathbb{E} \int_0^T |Z_s^n|^2 ds & \\ + \mathbb{E} \int_0^T \int_E |U_s^n(e)|^2 \lambda(de) ds + \mathbb{E}|K_T^n|^2 & \leq \quad C. \end{aligned} \tag{1.3.10}$$

Finally, by writing from (1.3.1) that

$$\begin{aligned} \sup_{0 \leq t \leq T} |Y_t^n| \quad \leq \quad & |g(X_T)| + \int_0^T |f(X_s, Y_s, Z_s)| ds + K_T^n + \sup_{s \in [0,T]} \left| \int_0^T \langle Z_s, dW_s \rangle \right| \\ & + \int_0^T \int_E |U_s^n(e) - c(X_{s^-}, Y_{s^-}, Z_s, e)| \mu(ds, de), \end{aligned}$$

we obtain the required result from the Burkholder-Davis-Gundy inequality, the linear growth condition on f, c, and (1.3.10). □

Remark 1.3.2 A closer look at the proof leading to the estimate in (1.3.7) shows that there exists a universal constant C, depending only on T, and the linear growth condition constants of f, c, such that for each $n \in \mathbb{N}$:

$$\begin{aligned} \sup_{t \in [0,T]} \mathbb{E}[Y_t^n]^2 \quad \leq \quad & C\Big(1 + \mathbb{E}|g(X_T)|^2 + \\ & \mathbb{E}\Big[\int_0^T |X_t|^2 dt \Big] + \mathbb{E}\Big[\sup_{t \in [0,T]} |\tilde{Y}_t|^2 \Big] \Big). \end{aligned} \tag{1.3.11}$$

Lemma 1.3.4 *Under* **(H2)** *(or* **(H1)** *in the case : $h(u,e) = -u$), the sequence of processes (Y_t^n) converges increasingly to a process (Y_t) with $\mathbb{E}\Big[\sup_{t \in [0,T]} |Y_t|^2 \Big] < \infty$. The convergence also holds in $\mathbf{L^2_{\mathbb{F}}(0,T)}$ and for every stopping time $\tau \in [0,T]$, the sequence of random variables (Y_τ^n) converges to Y_τ in $\mathbf{L^2(\Omega}, \mathcal{F}_\tau)$, i.e.*

$$\lim_{n \to \infty} \mathbb{E}\Big[\int_0^T |Y_t^n - Y_t|^2 dt \Big] = 0 \quad \textit{and} \quad \lim_{n \to \infty} \mathbb{E}\Big[|Y_\tau^n - Y_\tau|^2 \Big] = 0. \tag{1.3.12}$$

Proof. From Lemmas 1.3.1 and 1.3.2, the (nondecreasing) limit

$$Y_t \quad := \quad \lim_{n\to\infty} Y_t^n, \quad 0 \leq t \leq T, \tag{1.3.13}$$

exists almost surely, and this defines an adapted process Y. Moreover, by Lemma 1.3.3 and convergence monotone theorem, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big] \quad < \quad \infty.$$

From the dominated convergence theorem, we also get the convergences (1.3.12). □

We now focus on the convergence of the diffusion and jump components (Z^n, U^n). In our context, we cannot prove the strong convergence of (Z^n, U^n) in $\mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu})$, and so the strong convergence of $\int_0^t Z^n dW$ and $\int_0^t \int_E U^n(s,e)\mu(ds,de)$ in $\mathbf{L^2}(\mathbf{\Omega}, \mathcal{F}_\mathbf{t})$, see Remark 1.3.3. Instead, we follow and extend arguments of Peng [64], and we shall prove that (Z^n, U^n) converge in $\mathbf{L^p}(\mathbf{W}) \times \mathbf{L^p}(\tilde{\mu})$, for $1 \leq p < 2$. First, we show the following weak convergence and decomposition result.

Lemma 1.3.5 *Under* **(H2)** *(or* **(H1)** *in the case: $h(u,e) = -u$), there exist $\phi \in \mathbf{L^2_{\mathbb{F}}}(\mathbf{0},\mathbf{T})$, $Z \in \mathbf{L^2}(\mathbf{W})$, $V \in \mathbf{L^2}(\tilde{\mu})$ and K nondecreasing predictable with $\mathbb{E}[|K_T|^2] < \infty$, such that the limit Y in* (1.3.13) *has the form*

$$Y_t \quad = \quad Y_0 - \int_0^t \phi_s ds - K_t + \int_0^t \langle Z_s, dW_s\rangle + \int_0^t \int_E V_s(e)\mu(ds,de), \tag{1.3.14}$$

for all $t \in [0,T]$. Moreover, in the above decomposition of Y, the components Z and V are unique, and are respectively the weak limits of (Z^n) in $\mathbf{L^2}(\tilde{\mu})$ and of (V^n) in $\mathbf{L^2}(\tilde{\mu})$ where $V_t^n(e) = U_t^n(e) - c(X_{t^-}, Y_{t^-}^n, Z_t^n, e)$, ϕ is the weak limit in $\mathbf{L^2_{\mathbb{F}}}(\mathbf{0},\mathbf{T})$ of a subsequence of $(f^n) := (f(X,Y^n,Z^n))$, and K is the weak limit in $\mathbf{L^2_{\mathbb{F}}}(\mathbf{0},\mathbf{T})$ of a subsequence of (K^n). Consequently, the processes Y and K are càdlàg i.e. $Y \in \mathcal{S}^2$ and $K \in \mathbf{A}^2$.

Proof. By Lemma 1.3.3, and the linear growth conditions on f, c together with (1.2.2), the sequences (f^n), (Z^n), (V^n) are weakly compact, respectively in $\mathbf{L^2_{\mathbb{F}}}(\mathbf{0},\mathbf{T})$, $\mathbf{L^2}(\mathbf{W})$ and $\mathbf{L^2}(\tilde{\mu})$. Then, up to a subsequence, (f^n), (Z^n), (V^n) converge weakly to ϕ, Z and V. By Itô representation of martingales, we then get the following weak convergence in $\mathbf{L^2}(\mathbf{\Omega}, \mathcal{F}_\tau)$ for each stopping time $\tau \leq T$:

$$\int_0^\tau f_s^n ds \rightharpoonup \int_0^\tau \phi_s ds, \qquad \int_0^\tau \langle Z_s^n, dW_s\rangle \rightharpoonup \int_0^\tau \langle Z_s, dW_s\rangle,$$
$$\int_0^\tau \int_E V_s^n(e)\mu(ds,de) \rightharpoonup \int_0^\tau \int_E V_s(e)\mu(ds,de).$$

Since, we have from (1.3.1):

$$\begin{aligned} K_\tau^n \quad &= \quad -Y_\tau^n + Y_0^n - \int_0^\tau f_s^n ds \\ &\quad + \int_0^\tau \langle Z_s^n, dW_s\rangle + \int_0^\tau \int_E V_s^n(e)\mu(ds,de), \end{aligned} \tag{1.3.15}$$

we also have the weak convergence in $\mathbf{L^2}(\mathbf{\Omega}, \mathcal{F}_\tau)$:

$$\begin{aligned} K^n_\tau \rightharpoonup K_\tau \quad := \quad & -Y_\tau + Y_0 - \int_0^\tau \phi_s ds \\ & + \int_0^\tau \langle Z_s, dW_s\rangle + \int_0^\tau \int_E V_s(e)\mu(ds, de). \end{aligned} \tag{1.3.16}$$

From Lemma 2.2 in [64], (1.3.13) and (1.3.16) we obtain the càdlàg reguarity of Y. The process K inherits from K^n the nondecreasing path property, is square integrable and adapted from (1.3.16).Moreover, by dominated convergence theorem, we see that K^n converges weakly to K in $\mathbf{L^2}(\mathbf{0}, \mathbf{T})$. Since K^n is continuous, and so predictable, we deduce that K is also predictable, and we obtain the decomposition (1.3.14) for Y. The uniqueness of Z follows by identifying the Brownian parts and finite variation parts, and the uniqueness of V is then obtained by identifying the predictable parts and by recalling that the jumps of μ are inaccessible. We conclude that (Z, V) is uniquely determined in (1.3.14), and thus the whole sequence (Z^n, V^n) converges weakly to (Z, V) in $\mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu})$.

The càdlàg regularity of the processes Y and K is a consequence of the decomposition (1.3.14) of Y given by and the following Lemma proved in [64]. □

Lemma 1.3.6 *Let $(y^n)_n$ be a sequence of (deterministic) càdlàg processes defined on $[0, T]$ that increasingly converges to y: for each $t \in [0, T]$, $y^n_t \uparrow x_t$, with $y_t = b_t - k_t$, where b is an càdlàg process and k is an nondecreasing process with $k_0 = 0$ and $k_T < \infty$. Then y and k are also càdlàg processes.*

The sequence (U^n) is bounded in $\mathbf{L^2}(\tilde{\mu})$, and so, up to a subsequence, converges weakly to some $U \in \mathbf{L^2}(\tilde{\mu})$. The next step is to show that the whole sequence (U^n) converges to U and to identify in the decomposition (1.3.14) ϕ_t with $f(X_t, Y_t, Z_t)$, and $V_t(e)$ with $U_t(e)-c(X_{t^-}, Y_{t^-}, Z_t, e)$. Since f and c are nonlinear, we need a result of strong convergence for (Z^n) and (U^n) to enable us to pass the limit in $f(X_t, Y^n_t, Z^n_t)$ as well as in $U^n_t(e) - c(X_{t^-}, Y^n_{t^-}, Z^n_t, e)$, and to eventually prove the convergence of the penalized BSDEs to the minimal solution of our jump-constrained BSDE. We shall borrow a useful technique of Peng [64] to carry out this task.

Theorem 1.3.1 *Under* **(H2)**, *there exists a unique minimal solution $(Y, Z, U, K) \in \mathcal{S}^\mathbf{2} \times \mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$ with K predictable, to (1.2.10)-(1.2.11). Y is the increasing limit of (Y^n) in (1.3.13) and also in $\mathbf{L^2_{\mathbb{F}}}(\mathbf{0}, \mathbf{T})$, K is the weak limit of (K^n) in $\mathbf{L^2_{\mathbb{F}}}(\mathbf{0}, \mathbf{T})$, and for any $p \in [1, 2)$,*

$$\|Z^n - Z\|_{\mathbf{L^p(W)}} + \|U^n - U\|_{\mathbf{L^p}(\tilde{\mu})} \quad \longrightarrow \quad 0,$$

as n goes to infinity. Moreover, in the case : $h(u, e) = -u$, $(Y, Z, \bar{K})$ is the unique minimal solution to (1.2.12) with $\bar{K}_t = K_t - \int_0^t \int_E U_s(e)\mu(ds, de)$, and this holds true under **(H1)**. *Consequently, the minimal solution Y to (1.2.12) and to (1.2.10)-(1.2.13) are the same.*

A similar result was proved by Royer [73] in a different context. His applies to the Doob-Meyer decomposition of nonlinear $f-$supermartingales with jumps.

Proof. We apply Itô's formula to $|Y_t^n - Y_t|^2$ on a subinterval $(\sigma, \tau]$, with $0 \leq \sigma < \tau \leq T$, two stopping times. Recall the decomposition (1.3.14), (1.3.15) of Y, Y^n, and observe that K^n is continuous, and $\Delta(Y_t^n - Y_t) = \Delta K_t + \int_E (V_t^n(e) - V_t(e))\mu(\{t\}, de)$. We then have :

$$
\begin{aligned}
\mathbb{E}|Y_\tau^n - Y_\tau|^2 &= \mathbb{E}|Y_\sigma^n - Y_\sigma|^2 + \mathbb{E}\int_\sigma^\tau |Z_s^n - Z_s|^2 ds + 2\mathbb{E}\int_\sigma^\tau [Y_s^n - Y_s][\phi_s - f_s^n]ds \\
&\quad - 2\mathbb{E}\int_\sigma^\tau [Y_s^n - Y_s]dK_s^n + 2\mathbb{E}\int_{(\sigma,\tau]} [Y_{s^-}^n - Y_{s^-}]dK_s + \mathbb{E}\sum_{t\in(\sigma,\tau]} |\Delta K_t|^2 \\
&\quad + \mathbb{E}\int_{(\sigma,\tau]}\int_E [|Y_{s^-}^n - Y_{s^-} + V_s^n(e) - V_s(e)|^2 - |Y_{s^-}^n - Y_{s^-}|^2]\mu(ds, de) \\
&= \mathbb{E}|Y_\sigma^n - Y_\sigma|^2 + \mathbb{E}\int_\sigma^\tau |Z_s^n - Z_s|^2 ds + 2\mathbb{E}\int_\sigma^\tau [Y_s^n - Y_s][\phi_s - f_s^n]ds \\
&\quad - 2\mathbb{E}\int_\sigma^\tau [Y_s^n - Y_s]dK_s^n \; + \; 2\mathbb{E}\int_{(\sigma,\tau]} [Y_{s^-}^n - Y_{s^-} + \Delta K_s]dK_s \\
&\quad - \mathbb{E}\sum_{t\in(\sigma,\tau]} |\Delta K_t|^2 + \mathbb{E}\int_\sigma^\tau \int_E |V_s^n(e) - V_s(e)|^2 \lambda(de)ds \\
&\quad + 2\mathbb{E}\int_\sigma^\tau \int_E (Y_s^n - Y_s)(V_s^n(e) - V_s(e))\lambda(de)ds.
\end{aligned}
$$

Since $(Y_s^n - Y_s)dK_s^n \leq 0$, and by using the inequality $2ab \geq -\frac{a^2}{2} - 2b^2$ with $a = V_s^n(e) - V_s(e)$ and $b = Y_s^n - Y_s$, we obtain :

$$
\begin{aligned}
&\mathbb{E}\int_\sigma^\tau |Z_s^n - Z_s|^2 ds + \frac{1}{2}\mathbb{E}\int_\sigma^\tau \int_E |V_s^n(e) - V_s(e)|^2 \lambda(de)ds \\
\leq\; &\mathbb{E}|Y_\tau^n - Y_\tau|^2 + 2\mathbb{E}\int_\sigma^\tau |Y_s^n - Y_s|^2 ds + 2\mathbb{E}\int_\sigma^\tau |Y_s^n - Y_s||\phi_s - f_s^n|ds \\
&+ 2\mathbb{E}\int_{(\sigma,\tau]} |Y_{s^-}^n - Y_{s^-} + \Delta K_s|dK_s + \mathbb{E}\sum_{t\in(\sigma,\tau]} |\Delta K_t|^2.
\end{aligned}
\tag{1.3.17}
$$

The first two terms of the right side of (1.3.17) converge to zero by (1.3.12) in Lemma 1.3.4. The third term also tends to zero since $(\phi - f^n)_n$ is bounded in $\mathbf{L^2(0, T)}$, and so by Cauchy-Schwarz inequality:

$$
\mathbb{E}\int_0^T |Y_s^n - Y_s||\phi_s - f_s^n|ds \;\leq\; C\Big(\mathbb{E}\int_0^T |Y_s^n - Y_s|^2 ds\Big)^{\frac{1}{2}} \;\to\; 0. \tag{1.3.18}
$$

For the fourth term, we notice that the jumps of Y^n are inaccessible since they are determined by the Poisson random measure μ. Thus, the predictable projection of Y^n is ${}^pY_t^n = Y_{t^-}^n$. Similarly, from (1.3.14), and since K is predictable, we see that ${}^pY_t = Y_{t^-} - \Delta K_t$. Since Y^n increasingly converges to Y, then ${}^pY^n$ also increasingly converges to ${}^pY$, and by the dominated convergence theorem, we obtain:

$$\lim_{n\to\infty} \mathbb{E}\int_{(0,T]} |Y^n_{s^-} - Y_{s^-} + \Delta K_s| dK_s \;=\; 0. \tag{1.3.19}$$

For the last term in (1.3.17), we apply Lemma 2.3 in [64] to the predictable nondecreasing process K: for any $\delta, \varepsilon > 0$, there exists a finite number of pairs of stopping times (σ_k, τ_k), $k = 0, \ldots, N$, with $0 < \sigma_k \leq \tau_k \leq T$, such that all the intervals $(\sigma_k, \tau_k]$ are disjoint and

$$\mathbb{E}\sum_{k=0}^{N}(\tau_k - \sigma_k) \geq T - \frac{\varepsilon}{2}, \qquad \mathbb{E}\sum_{k=0}^{N}\sum_{\sigma_k < t \leq \tau_k}(\Delta K_t)^2 \leq \frac{\varepsilon\delta}{3}. \tag{1.3.20}$$

We should note that in [64] the filtration is Brownian, therefore it is continuous, and hence each stopping time σ_k can be approximated by a sequence of announceable stopping times In our case the stopping times σ_k's are constructed as the successive times of jumps of the predictable process K with size bigger than some given positive level, the approximation of σ_k by announceable stopping times is again possible. We can thus argue exactly the same way as in Lemma 2.3 in [64] to derive both estimates in (1.3.20).

We now apply estimate (1.3.17) for each $\sigma = \sigma_k$ and $\tau = \tau_k$, and then take the sum over $k = 0, \ldots, N$. It follows that

$$\begin{aligned}
&\sum_{k=0}^{N}\mathbb{E}\int_{\sigma_k}^{\tau_k}|Z^n_s - Z_s|^2 ds + \frac{1}{2}\sum_{k=0}^{N}\mathbb{E}\int_{\sigma_k}^{\tau_k}\int_E |V^n_s(e) - V_s(e)|^2\lambda(de)ds \\
\leq\;& \sum_{k=0}^{N}\mathbb{E}|Y^n_{\tau_k} - Y_{\tau_k}|^2 + 2\mathbb{E}\int_0^T |Y^n_s - Y_s|^2 ds + 2\mathbb{E}\int_0^T |Y^n_s - Y_s||\phi_s - f^n_s| ds \\
&+ 2\mathbb{E}\int_{(0,T]}|Y^n_{s^-} - Y_{s^-} + \Delta K_s| dK_s + \sum_{k=0}^{N}\mathbb{E}\sum_{t\in(\sigma_k,\tau_k]}|\Delta K_t|^2.
\end{aligned}$$

From the convergence results in Lemma 1.3.4, (1.3.18) and (1.3.19), we deduce that

$$\begin{aligned}
&\limsup_{n\to\infty}\sum_{k=0}^{N}\mathbb{E}\int_{\sigma_k}^{\tau_k}|Z^n_s - Z_s|^2 ds + \frac{1}{2}\sum_{k=0}^{N}\mathbb{E}\int_{\sigma_k}^{\tau_k}\int_E |V^n_s(e) - V_s(e)|^2\lambda(de)ds \\
\leq\;& \sum_{k=0}^{N}\mathbb{E}\sum_{t\in(\sigma_k,\tau_k]}|\Delta K_t|^2 \;\leq\; \frac{\varepsilon\delta}{3}.
\end{aligned}$$

Thus, there exists an integer $\ell_{\varepsilon\delta} > 0$ such that for all $n \geq \ell_{\varepsilon\delta}$, we have

$$\sum_{k=0}^{N}\mathbb{E}\int_{\sigma_k}^{\tau_k}|Z^n_s - Z_s|^2 ds + \frac{1}{2}\sum_{k=0}^{N}\mathbb{E}\int_{\sigma_k}^{\tau_k}\int_E |V^n_s(e) - V_s(e)|^2\lambda(de)ds \;\leq\; \frac{\varepsilon\delta}{2}.$$

This implies

$$dt\otimes\mathbb{P}\Big[(s,\omega)\in\bigcup_{k=0}^{N}(\sigma_k(\omega),\tau_k(\omega)]\times\Omega \;:\, |Z^n_s(\omega) - Z_s(\omega)|^2 \geq \delta\Big] \;\leq\; \frac{\varepsilon}{2},$$

and

$$dt \otimes \lambda \otimes \mathbb{P}\Big[(s,e,\omega) \in \bigcup_{k=0}^{N} (\sigma_k(\omega), \tau_k(\omega)] \times \Omega \times E \ : |V_s^n(e,\omega) - V_s(e,\omega)|^2 \geq \delta\Big] \quad \leq \quad \varepsilon.$$

Together with (1.3.20), it follows that

$$dt \otimes \mathbb{P}\left[(s,\omega) \in [0,T] \times \Omega \ : |Z_s^n(\omega) - Z_s(\omega)|^2 \geq \delta\right] \quad \leq \quad \varepsilon,$$

and

$$dt \otimes \lambda \times \mathbb{P}\left[(s,e,\omega) \in [0,T] \times E \times \Omega \ : |V_s^n(e,\omega) - V_s(e,\omega)|^2 \geq \delta\right] \quad \leq \quad \varepsilon(1+\lambda(E)).$$

We deduce that for all $\delta > 0$,

$$\lim_{n\to\infty} dt \otimes \mathbb{P}\left[(s,\omega) \in [0,T] \times \Omega \ : |Z_s^n(\omega) - Z_s(\omega)|^2 \geq \delta\right] \quad = \quad 0$$

and

$$\lim_{n\to\infty} dt \otimes \lambda \otimes \mathbb{P}\left[(s,e,\omega) \in [0,T] \times E \times \Omega \ : |V_s^n(e,\omega) - V_s(e,\omega)|^2 \geq \delta\right] \quad = \quad 0.$$

This means that the sequences $(Z^n)_n$ and $(V^n)_n$ converge in measure respectively to Z and V. Since they are bounded respectively in $\mathbf{L^2(W)}$ and $\mathbf{L^2}(\tilde{\mu})$, they are uniformly integrable in $\mathbf{L^p(W)}$ and $\mathbf{L^p}(\tilde{\mu})$ for any $p \in [1,2)$, respectively. Thus, (Z^n) and (V^n) converge strongly to Z and V in $\mathbf{L^p(W)}$ and $\mathbf{L^p}(\tilde{\mu})$, respectively. Recalling that $U_t^n(e) = V_t^n(e) + c(X_{t^-}, Y_{t^-}^n, Z_t^n, e)$, and by the Lipschitz condition on c, we deduce that the sequence (U^n) converges strongly in $\mathbf{L^p}(\tilde{\mu})$, for $p \in [1,2)$, to U defined by :

$$U_t(e) \quad = \quad V_t(e) + c(X_{t^-}, Y_{t^-}, Z_t, e), \quad 0 \leq t \leq T, \ e \in E.$$

By the Lipschitz condition on f, we also have the strong convergence in $\mathbf{L^p_{\mathbb{F}}(0,T)}$ of (f^n) $= (f(X, Y^n, Z^n))$ to $f(X,Y,Z)$. Since ϕ is the weak limit of (f^n) in $\mathbf{L^2_{\mathbb{F}}(0,T)}$, we deduce that $\phi = f(X,Y,Z)$. Therefore, with the decomposition (1.3.14) and since $Y_T = \lim_n Y_T^n$ $= g(X_T)$, we obtain immediately that (Y,Z,U,K) satisfies the BSDE (1.2.10). Moreover, from the strong convergence in $\mathbf{L^1}(\tilde{\mu})$ of (U^n) to U, and the Lipschitz condition on h, we have

$$\mathbb{E}\int_0^T \int_E h^-(U_s^n(e),e)\lambda(de)ds \quad \to \quad \mathbb{E}\int_0^T \int_E h^-(U_s(e),e)\lambda(de)ds,$$

as n goes to infinity. Since $K_T^n = n\int_0^T \int_E h^-(U_s^n(e),e)\lambda(de)ds$ is bounded in $\mathbf{L^2}(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}_T})$, this implies

$$\mathbb{E}\int_0^T \int_E h^-(U_s(e),e)\lambda(de)ds \quad = \quad 0,$$

and so the constraint (1.2.11) is satisfied. Hence, (Y, Z, K, U) is a solution to the constrained BSDE (1.2.10)-(1.2.11), and by Lemma 1.3.2, $Y = \lim Y^n$ is the minimal solution. The uniqueness of Z follows by identifying the Brownian parts and the finite variation parts, and then the uniqueness of (U, K) is obtained by identifying the predictable parts and by recalling that the jumps of μ are inaccessible.

Finally, in the case $h(u, e) = -u$, the process

$$\bar{K}_t = K_t - \int_0^t \int_E U_s(e)\mu(ds, de), \quad 0 \leq t \leq T,$$

lies in $\mathbf{A^2}$, and the triple $(Y, Z, \bar{K})$ is solution to (1.2.12). Again, by Lemma 1.3.2, this shows that Y is the minimal solution to (1.2.10) and to (1.2.12). The uniqueness of $(Y, Z, \bar{K})$ is immediate by identifying the Brownian part and the finite variation part. □

Remark 1.3.3 From the estimate (1.3.17), it is clear that once the process K is continuous, i.e. $\Delta K_t = 0$, then (Z^n, U^n) converges strongly to (Z, U) in $\mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$. This occurs in reflected BSDE's as in [29] or [40], see also Remark 1.4.3. In the case of constraints on jump component U as in (1.2.10)-(1.2.11), the situation is more complicated, and the process K is in general only predictable. The same feature also occurs for constraints on Z as in [64]. To overcome this difficulty, we use the estimations (1.3.20) of the contribution of the jumps of K, which allow to obtain the strong convergence of (Z^n, U^n) in $\mathbf{L^p(W)} \times \mathbf{L^p}(\tilde{\mu})$ for $p \in [1, 2)$. Finally, notice that for the minimal solution $(Y, Z, \tilde{K})$ to the BSDE (1.2.12), the process $\tilde{K}$ is not predictable.

1.3.3 The case of impulse control

In the impulse control case (i.e. f and c depend only on X and $h(u, e) = -u$), we have seen in Theorem 1.2.1 that the minimal solution to our constrained BSDE has the following functional explicit representation :

$$Y_t = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^\nu \Big[g(X_T) + \int_t^T f(X_s)ds + \int_t^T \int_E c(X_{s^-}, e)\mu(ds, de) \Big| \mathcal{F}_t \Big].$$

In this case, we also have a functional explicit representation of the solution Y^n to the penalized BSDE (1.3.1) :

$$\begin{aligned} Y_t^n = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}_n} \mathbb{E}^\nu \Big[& g(X_T) + \int_t^T f(X_s)ds \\ & + \int_t^T \int_E c(X_{s^-})\mu(ds, de) \Big| \mathcal{F}_t \Big], \end{aligned} \tag{1.3.21}$$

where $\mathcal{V}_n = \{\nu \in \mathcal{V} \;;\; \nu_s(e) \leq n \; \forall (s, e) \in [0, T] \times E \; a.s.\}$. Indeed, denote by $\bar{Y}^n$ the right side of (1.3.21). By writing that (Y^n, Z^n, U^n) is the solution of the penalized BSDE (1.3.1), taking the expectation under $\mathbb{P}^\nu$, for $\nu \in \mathcal{V}_n$, and recalling that W is a $\mathbb{P}^\nu$-Brownian motion, and $\nu\lambda(de)$ is the intensity measure of μ under $\mathbb{P}^\nu$, we obtain :

$$\begin{aligned} Y_t^n &= \mathbb{E}^\nu\Big[g(X_T) + \int_t^T f(X_s)ds + \int_t^T \int_E c(X_{s^-}, e)\mu(ds, de)\Big|\mathcal{F}_t\Big] \\ &\quad + \mathbb{E}^\nu\Big[\int_t^T \int_E \{n[U_s^n(e)]_+ - \nu_s(e)U_s^n(e)\}\,\lambda(de)ds\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{1.3.22}$$

Since this equality holds for any $\nu \in \mathcal{V}_n$, and observing that $n[U_s^n(e)]_+ - \nu_s(e)U_s^n(e) \geq 0$, for all $\nu \in \mathcal{V}_n$, we have

$$\begin{aligned} \bar{Y}_t^n \;\leq\; Y_t^n \;&\leq\; \tilde{Y}_t^n \\ &+\mathbb{E}^\nu\Big[\int_t^T \int_E \{n[U_s^n(e)]_+ - \nu_s(e)U_s^n(e)\}\,\lambda(de)ds\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{1.3.23}$$

Let us now consider the family $(\nu^\varepsilon)_\varepsilon$ of $\mathcal{V}_n$ defined by

$$\nu_s^\varepsilon(e) = \begin{cases} n & \text{if } U_s^n(e) > 0 \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then, by using the same argument as in the proof of Lemma 1.3.2, we show that

$$\mathbb{E}^{\nu^\varepsilon}\Big[\int_t^T \int_E \{n[U_s^n(e)]_+ - \nu_s(e)U_s^n(e)\}\,\lambda(de)ds\Big|\mathcal{F}_t\Big] \;\to\; 0 \text{ as } \varepsilon \to 0,$$

which proves with (1.3.23) that $Y_t^n = \bar{Y}_t^n$.

The representation (1.3.21) has a nice interpretation. It means that the value function of an impulse control problem can be approximated by the value function of the same impulse control problem but with strategies whose numbers of orders are bounded *on average* by $nT\lambda(E)$. This has to be compared with the classical approximation by iterated optimal stopping problems, where the n-th iteration corresponds to the value of the same impulse control problem but where the number of orders is smaller than n. The numerical advantage of the penalized approximation is that it does not require iterations.

1.4 Relation with quasi-variational inequalities

In this section, we show that minimal solutions to the jump-constrained BSDEs provide a probabilistic representation of solutions to parabolic QVIs of the form:

$$\min\Big[-\frac{\partial v}{\partial t} - \mathcal{L}v - f(\cdot, v, \sigma^\intercal D_x v)\,,\, \inf_{e\in E} h(\mathcal{H}^e v - v, e)\Big] = 0, \quad \text{on } [0,T) \times \mathbb{R}^d \tag{1.4.1}$$

where $\mathcal{L}$ is the second order local operator

$$\mathcal{L}v(t,x) = \langle b(x), D_x v(t,x)\rangle + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x)D_x^2 v(t,x)),$$

and $\mathcal{H}^e$, $e \in E$, are the nonlocal operators

$$\mathcal{H}^e v(t,x) = v(t, x+\gamma(x,e)) + c(x, v(t,x), \sigma^{\intercal}(x) D_x v(t,x), e).$$

For such nonlocal operators, we denote for $q \in \mathbb{R}^d$:

$$\mathcal{H}^e[t,x,q,v] \quad = \quad v(t, x+\gamma(x,e)) + c(x, v(t,x), \sigma^{\intercal}(x) q, e).$$

Note that when $h(u)$ does not depend on e, and since it is nonincreasing in u, the QVI (1.4.1) may be written equivalently in

$$\min\Big[-\frac{\partial v}{\partial t} - \mathcal{L}v - f(\cdot, v, \sigma^{\intercal} D_x v) \ , \ h(\mathcal{H}v - v) \Big] \quad = \quad 0, \qquad \text{on } [0,T) \times \mathbb{R}^d,$$

with $\mathcal{H}v = \sup_{e\in E} \mathcal{H}^e v$. In particular, this includes the case of QVI associated to impulse controls for $h(u) = -u$, and f, c independent of y, z.

We shall use the penalized parabolic integral partial differential equation (IPDE) associated to the penalized BSDE (1.3.1), for each $n \in \mathbb{N}$:

$$\begin{aligned} -\frac{\partial v_n}{\partial t} &- \mathcal{L}v_n - f(\cdot, v_n, \sigma^{\intercal} D_x v_n) \\ &-n \int_E h^-(\mathcal{H}^e v_n - v_n, e)\lambda(de) \quad = \quad 0, \qquad \text{on} \ \ [0,T) \times \mathbb{R}^d. \end{aligned} \tag{1.4.2}$$

To complete the PDE characterization of the function v, we need to provide a suitable boundary condition. In general, we can not expect to have $v(T^-,.) = g$, and we shall consider the relaxed boundary condition given by the equation:

$$\min\Big[v(T^-,\cdot) - g \ , \ \inf_{e\in E} h(\mathcal{H}^e v(T^-,\cdot) - v(T^-,\cdot), e) \Big] \quad = \quad 0, \ \text{on} \ \ \mathbb{R}^d, \tag{1.4.3}$$

In the sequel, we shall assume in addition to the conditions of paragraph 1.2.1 that the functions γ, f, c, and h are continuous with respect to all their arguments.

1.4.1 Viscosity properties

Solutions of (1.4.1), (1.4.2) and (1.4.3) are considered in the (discontinuous) viscosity sense, and it will be convenient in the sequel to define the notion of viscosity solutions in terms of sub- and super-jets. We refer to [45], [78] and more recently to the book [60] for the notion of viscosity solutions to QVIs. For a locally bounded function u on $[0,T] \times \mathbb{R}^d$, we define its lower semicontinuous (lsc in short) u_*, and upper semicontinuous (usc in short) envelope u^* by

$$u_*(t,x) \;=\; \liminf_{(t',x')\to(t,x), t'<T} u(t',x'), \qquad u^*(t,x) \;=\; \limsup_{(t',x')\to(t,x), t'<T} u(t',x').$$

Definition 1.4.1 *(Subjets and superjets)*
(i) For a function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$*, lsc (resp. usc), we denote by* $J^- u(t,x)$ *the parabolic subjet (resp.* $J^+ u(t,x)$ *the parabolic superjet) of* u *at* $(t,x) \in [0,T] \times \mathbb{R}^d$*, as the set of triples* $(p,q,M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ *satisfying*

$$\begin{aligned} u(t',x') \quad \geq \ (\textit{resp.} \leq) \quad & u(t,x) + p(t'-t) + \langle q, x'-x \rangle + \frac{1}{2} \langle x'-x, M(x'-x) \rangle \\ & + o(|t'-t| + |x'-x|^2). \end{aligned}$$

(ii) For a function $u : [0,T) \times \mathbb{R}^d \to \mathbb{R}$*, lsc (resp. usc), we denote by* $\bar{J}^- u(t,x)$ *the parabolic limiting subjet (resp.* $\bar{J}^+ u(t,x)$ *the parabolic limiting superjet) of* u *at* $(t,x) \in [0,T] \times \mathbb{R}^d$*, as the set of triples* $(p,q,M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ *such that*

$$\begin{aligned} (p,q,M) \ &= \ \lim_n (p_n, q_n, M_n), \qquad & (t,x) \ &= \ \lim_n (t_n, x_n), \\ \textit{with} \quad (p_n, q_n, M_n) \in J^- u(t_n, x_n) \ (\textit{resp.} \ J^+ u(t_n, x_n)), \qquad & u(t,x) \ &= \ \lim_n u(t_n, x_n). \end{aligned}$$

We now give the definition of viscosity solutions to (1.4.1), (1.4.2) and (1.4.3).

Definition 1.4.2 *(Viscosity solutions to* (1.4.1)*)*
(i) A function u*, lsc (resp. usc) on* $[0,T) \times \mathbb{R}^d$*, is called a viscosity supersolution (resp. subsolution) to* (1.4.1) *if for each* $(t,x) \in [0,T) \times \mathbb{R}^d$*, and any* $(p,q,M) \in \bar{J}^- u(t,x)$ *(resp.* $\bar{J}^+ u(t,x)$*), we have*

$$\begin{aligned} \min \Big[- p - \langle b(x), q \rangle - \frac{1}{2} \mathrm{tr}(\sigma\sigma^{\intercal}(x) M) - f(x, u(t,x), \sigma^{\intercal}(x) q) \ , \\ \inf_{e \in E} h(\mathcal{H}^e[t,x,q,u] - u(t,x), e) \Big] \quad \geq (\ \textit{resp.} \ \leq) \quad 0. \end{aligned}$$

(ii) A locally bounded function on $[0,T) \times \mathbb{R}^d$ *is called a viscosity solution to* (1.4.1) *if* u_* *is a viscosity supersolution and* u^* *is a viscosity subsolution to* (1.4.1).

Definition 1.4.3 *(Viscosity solutions to* (1.4.2)*)*
(i) A function u*, lsc (resp. usc) on* $[0,T) \times \mathbb{R}^d$*, is called a viscosity supersolution (resp. subsolution) to* (1.4.2) *if for each* $(t,x) \in [0,T) \times \mathbb{R}^d$*, and any* $(p,q,M) \in \bar{J}^- u(t,x)$ *(resp.* $\bar{J}^+ u(t,x)$*), we have*

$$\begin{aligned} -p - \langle b(x), q \rangle - \frac{1}{2} \mathrm{tr}(\sigma\sigma^{\intercal}(x) M) - f(x, u(t,x), \sigma^{\intercal}(x) q) \\ -n \int_E h^-(\mathcal{H}^e[t,x,q,u] - u(t,x), e) \lambda(de) \quad \geq (\ \textit{resp.} \ \leq) \quad 0. \end{aligned}$$

(ii) A locally bounded function u *on* $[0,T) \times \mathbb{R}^d$ *is called a viscosity solution to* (1.4.2) *if* u_* *is a viscosity supersolution and* u^* *is a viscosity subsolution to* (1.4.2).

Definition 1.4.4 *(Viscosity solutions to* (1.4.3)*)*
(i) A function u*, lsc (resp. usc) on* $[0,T] \times \mathbb{R}^d$*, is called a viscosity supersolution (resp. subsolution) to* (1.4.3) *if for each* $x \in \mathbb{R}^d$*, and any* $(p,q,M) \in \bar{J}^- u(T,x)$ *(resp.* $\bar{J}^+ u(T,x)$*), we have*

$$\min \Big[u(T,x) - g(x), \ \inf_{e \in E} h(\mathcal{H}^e[T,x,q,u] - u(T,x), e) \Big] \quad \geq (\ \textit{resp.} \ \leq) \quad 0.$$

(ii) A locally bounded function u on $[0,T]\times\mathbb{R}^d$ is called a viscosity solution to (1.4.3) *if u_* is a viscosity supersolution and u^* is a viscosity subsolution to* (1.4.3).

Remark 1.4.1 An equivalent definition of viscosity super and subsolution to (1.4.3), which shall be used later, is the following in terms of test functions : a function u, lsc (resp. usc) on $[0,T]\times\mathbb{R}^d$, is called a viscosity supersolution (resp. subsolution) to (1.4.3) if for each $(t,x)\in[0,T)\times\mathbb{R}^d$, and any $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that (t,x) is a minimum (resp. maximum) global of $u-\varphi$, we have

$$\min\Big[u(T,x)-g(x),\ \inf_{e\in E} h(\mathcal{H}^e[T,x,D_x\varphi(T,x),u]-u(T,x),e)\Big] \quad \geq (\text{ resp. } \leq) \quad 0.$$

We have similar equivalent definitions of viscosity super and subsolution to (1.4.1) in terms of test functions.

We slightly strengthen Assumption **(H1)** or **(H2)** by

(H1') There exists a quadruple $(\tilde{Y},\tilde{Z},\tilde{K})\in\mathcal{S}^{\mathbf{2}}\times\mathbf{L^2(W)}\times\mathbf{A^2}$ satisfying (1.2.12), with $\tilde{Y}_t = \tilde{v}(t,X_t)$, $0\leq t\leq T$, for some function deterministic $\tilde{v}$ satisfying a linear growth condition

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{|\tilde{v}(t,x)|}{1+|x|} < +\infty$$

(H2') There exists a quadruple $(\tilde{Y},\tilde{Z},\tilde{K},\tilde{U})\in\mathcal{S}^{\mathbf{2}}\times\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{A^2}$ satisfying (1.2.10)-(1.2.11), with $\tilde{Y}_t=\tilde{v}(t,X_t)$, $0\leq t\leq T$, for some function deterministic $\tilde{v}$ satisfying a linear growth condition

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{|\tilde{v}(t,x)|}{1+|x|} < +\infty$$

Under assumption **(H1')** (resp. **(H2')**), there esists for each $(t,x)\in[0,T]\times\mathbb{R}^d$ a unique minimal solution $\{(Y_s^{t,x},Z_s^{t,x},U_s^{t,x},K_s^{t,x}),t\leq s\leq T\}$ to (1.2.10)-(1.2.11) (resp. (1.2.12)-(1.2.13)) with $X=\{X_s^{t,x},t\leq s\leq T\}$, the solution to (1.2.1) starting from x at time t. We can then define the (deterministic) function $v:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ by

$$v(t,x) \quad := \quad Y_t^{t,x}, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \tag{1.4.4}$$

Similarly, we define the function

$$v_n(t,x) \quad := \quad Y_t^{n,t,x}, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \tag{1.4.5}$$

where $\{(Y_s^{n,t,x},Z_s^{n,t,x},U_s^{n,t,x}(.)),t\leq s\leq T\}$ is the unique solution to (1.3.1) with $X_s=X_s^{t,x}$, $t\leq s\leq T$.

We first have the following identification.

Proposition 1.4.1 *The function v links the processes $Y^{t,x}$ and $X^{t,x}$ by the relation:*

$$Y_\theta^{t,x} \quad = \quad v(\theta,X_\theta^{t,x}), \quad \textit{for all stopping time } \theta \textit{ valued in } [t,T]. \tag{1.4.6}$$

Proof. From the Markov property of the jump-diffusion process X, and uniqueness of a solution Y^n to the BSDE (1.3.1), we have (see e.g. [5])

$$Y_s^{t,x,n} = v_n\left(s, X_s^{t,x}\right), \quad t \leq s \leq T. \tag{1.4.7}$$

From Section 3, we know that v is the pointwise limit of v_n. Moreover, by (1.3.12), $Y_\theta^{t,x,n}$ converges to $Y_\theta^{t,x}$ as n goes to infinity, for all stopping time θ valued in $[t,T]$. We then obtain the required relation by passing to the limit in (1.4.7). □

Remark 1.4.2 Assumption **(H2')** (or **(H1')** which is weaker than **(H2')** in the case $h(u,e)$ $= -u$) ensures that the function v in (1.4.4) satisfies a linear growth condition, and is in particular locally bounded. Indeed, from (1.3.11) and by passing to the limit by Fatou's lemma for $v(t,x) = Y_t^{t,x} = \lim Y_t^{n,t,x}$, we have

$$\begin{aligned}\sup_{t\in[0,T]} |v(t,x)|^2 \;&\leq\; C\Big(1 + \mathbb{E}|g(X_T^{t,x})|^2 + \mathbb{E}\Big[\int_t^T |X_s^{t,x}|^2 dt\Big] \\ &\quad + \mathbb{E}\big[\sup_{s\in[t,T]} |\tilde{v}(s, X_s^{t,x})|^2\big]\Big).\end{aligned}$$

The result follows from the standard estimate

$$\mathbb{E}[\sup_{t\leq s\leq T} |X_s^{t,x}|^2] \;\leq\; C(1 + |x|^2),$$

and the linear growth conditions on g and $\tilde{v}$.

The relation between the penalized BSDE (1.3.1) and the penalized IPDE (1.4.2) is well-known from the results of [5]. Although our framework does not fit exactly into the one of [5], by mimicking closely the arguments in this paper and using comparison theorem in [73], we obtain the following result.

Proposition 1.4.2 *The function v_n in* (1.4.5) *is a continuous viscosity solution to* (1.3.1).

By adapting stability arguments for viscosity solutions to our context, we now prove the viscosity property of the function v to (1.4.1). We shall assume that the support of λ is the whole space E, i.e.

(HE) $\forall e \in E$, $\exists \mathcal{O}$ open neighborhood of e, s.t. $\lambda(\mathcal{O}) > 0$.

Theorem 1.4.1 *Under* **(H2')** *(or* **(H1')** *in the case : $h(u,e) = -u$), and* **(HE)***, the function v in* (1.4.4) *is a viscosity solution to* (1.4.1).

Proof. From the results of the previous section, we know that v is the pointwise limit of the nondecreasing sequence of functions (v_n). By continuity of v_n, we then have (see e.g.

[4] p. 91) :

$$
\begin{aligned}
v \;=\; v_* \;&=\; \lim_{n\to\infty} \inf{}_* v_n, && (1.4.8)\\
&\quad \text{where} \quad \lim_{n\to\infty} \inf{}_* v_n(t,x) \;:=\; \liminf_{\substack{n\,\to\,\infty\\ t'\,\to\, t,\, x'\,\to\, x}} v_n(t',x'),\\
v^* \;&=\; \lim_{n\to\infty} \sup{}^* v_n, && (1.4.9)\\
&\quad \text{where} \quad \lim_{n\to\infty} \sup{}^* v_n(t,x) \;:=\; \limsup_{\substack{n\,\to\,\infty\\ t'\,\to\, t,\, x'\,\to\, x}} v_n(t',x').
\end{aligned}
$$

(i) We first show the viscosity supersolution property for $v = v_*$. Let (t,x) be a point in $[0,T)\times\mathbb{R}^d$, and $(p,q,M)\in \bar{J}^- v(t,x)$. By (1.4.8) and Lemma 6.1 in [23], there exists sequences

$$
n_j \;\to\; \infty, \quad (p_j,q_j,M_j) \;\in\; J^- v_{n_j}(t_j,x_j),
$$

such that

$$
(t_j,x_j,v_{n_j}(t_j,x_j),p_j,q_j,M_j) \;\;\to\;\; (t,x,v(t,x),p,q,M). \tag{1.4.10}
$$

We also have by definition of $v = v_*$ and continuity of γ :

$$
v(t,x+\gamma(x,e)) \;\;\le\;\; \liminf_{j\to\infty} v_{n_j}(t_j,x_j+\gamma(x_j,e)), \quad \forall e\in E. \tag{1.4.11}
$$

Moreover, from the viscosity supersolution property for v_{n_j}, we have for all j

$$
\begin{aligned}
&-p_j - \langle b(x_j),q_j\rangle - \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x_j)M_j) - f(x_j,v_{n_j}(t_j,x_j),\sigma^{\intercal}(x_j)q_j)\\
&\qquad -n_j\int_E h^-(\mathcal{H}^e[t_j,x_j,q_j,v_{n_j}] - v_{n_j}(t_j,x_j),e)\lambda(de) \;\;\ge\;\; 0. \qquad (1.4.12)
\end{aligned}
$$

Let us check that the following inequality holds :

$$
\inf_{e\in E} h(\mathcal{H}^e[t,x,q,v] - v(t,x),e) \;\;\ge\;\; 0. \tag{1.4.13}
$$

We argue by contradiction, and assume there exists some $e_0\in E$ s.t.

$$
h(v(t,x+\gamma(x,e_0))+c(x,v(t,x),\sigma^{\intercal}(x)q,e_0) - v(t,x),e_0) \;\;<\;\; 0.
$$

Then, by continuity of σ, h, γ, c in all their variables, (1.4.10), (1.4.11), and the nonincreasing property of h, one may find some $\varepsilon>0$ and some open neighborhood $\mathcal{O}_0$ of e_0 such that for all j large enough :

$$
h(v_{n_j}(t_j,x_j+\gamma(x_j,e))+c(x_j,v_{n_j}(t_j,x_j),\sigma^{\intercal}(x_j)q_j,e) - v_{n_j}(t_j,x_j),e) \;\;\le\;\; -\varepsilon,
$$

for all $e\in\mathcal{O}_0$. Since the support of λ is E, this implies

$$
\int_E h^-(\mathcal{H}^e(t_j,x_j,q_j,v_{n_j}) - v_{n_j}(t_j,x_j),e)\lambda(de) \;\;\ge\;\; \varepsilon\lambda(\mathcal{O}_0) \;>\; 0.
$$

By sending j to infinity into (1.4.12), we get the required contradiction. On the other hand, by (1.4.12), we have

$$-p_j - \langle b(x_j), q_j \rangle - \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x_j)M_j) - f(x_j, v_{n_j}(t_j, x_j), \sigma^{\intercal}(x_j)q_j) \quad \geq \quad 0,$$

so that by sending j to infinity:

$$-p - \langle b(x), q \rangle - \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x)M) - f(x, v(t,x), \sigma^{\intercal}(x)q) \quad \geq \quad 0,$$

which proves, together with (1.4.13), that v is a viscosity supersolution to (1.4.1).
(ii) We conclude by showing the viscosity subsolution property for v^*. Let (t, x) a point in $[0, T) \times \mathbb{R}^d$, and $(p, q, M) \in \bar{J}^+ v^*(t, x)$ such that

$$\inf_{e \in E} h(\mathcal{H}^e[t, x, q, v^*] - v^*(t, x), e) \quad > \quad 0. \tag{1.4.14}$$

From (1.4.9) and Lemma 6.1 in [23], there exists sequences

$$n_j \ \to \ \infty, \quad (p_j, q_j, M_j) \ \in \ J^+ v_{n_j}(t_j, x_j),$$

such that

$$(t_j, x_j, v_{n_j}(t_j, x_j), p_j, q_j, M_j) \quad \to \quad (t, x, v^*(t, x), p, q, M). \tag{1.4.15}$$

By continuity of the functions c, γ, and definition of v^*, we also have

$$\limsup_{j \to \infty} \mathcal{H}^e[t_j, x_j, q_j, v_{n_j}] \quad \leq \quad \mathcal{H}^e[t, x, q, v^*], \quad \forall\, e \in E. \tag{1.4.16}$$

Now, from the viscosity subsolution property for v_{n_j}, we have for all j

$$\begin{aligned} &- p_j - \langle b(x_j), q_j \rangle - \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x_j)M_j) - f(x_j, v_{n_j}(t_j, x_j), \sigma^{\intercal}(x_j)q_j) \\ &\qquad -n_j \int_E h^-(\mathcal{H}^e[t_j, x_j, q_j, v_{n_j}] - v_{n_j}(t_j, x_j), e)\lambda(de) \quad \leq \quad 0. \end{aligned} \tag{1.4.17}$$

From (1.4.14)(which is uniform in $e \in E$)-(1.4.15)-(1.4.16), continuity assumptions on h, c, and the nonincreasing property of h, we have for j large enough

$$h(\mathcal{H}^e[t_j, x_j, q_j, v_{n_j}] - v_{n_j}(t_j, x_j), e) \quad > \quad 0, \quad \forall e \in E,$$

and so

$$\int_E h^-(\mathcal{H}^e[t_j, x_j, q_j, v_{n_j}] - v_{n_j}(t_j, x_j), e)\lambda(de) \quad = \quad 0.$$

Hence, by taking the limit as j goes to infinity, into (1.4.17), we conclude that

$$-p - \langle b(x), q \rangle - \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x)M) - f(x, v^*(t, x), \sigma^{\intercal}(x)q) \quad \leq \quad 0,$$

which shows the viscosity subsolution property for v^* to (1.4.1). □

We next turn to the boundary condition.

Theorem 1.4.2 *Under* **(H2')** *(or* **(H1')** *in the case : $h(u,e) = -u$), and* **(HE)**, *the function v in* (1.4.4) *is a viscosity solution to* (1.4.3).

In order to deal with the possible jump at the terminal condition, we need the following dynamic programming characterization of the minimal solution.

Lemma 1.4.1 *Let $(t,x) \in [0,T) \times \mathbb{R}^d$, and $(Y^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$ be a minimal solution to* (1.2.10)-(1.2.11) *on $[t,T]$ with $X_s = X_s^{t,x}$. Then for any stopping time θ valued in $[t,T]$, $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, K_s^{t,x})_{s\in[t,\theta]}$ is a minimal solution to :*

$$\begin{aligned} Y_s &= v(\theta, X_\theta^{t,x}) + \int_s^\theta f(X_r^{t,x}, Y_r, Z_r)dr + K_\theta^{t,x} - K_s^{t,x} - \int_s^\theta \langle Z_r, dW_r\rangle \\ &\quad - \int_s^\theta \int_E \Big(U_r(e) - c(X_{r^-}^{t,x}, Y_{r^-}, Z_r, e)\Big)\, \mu(dr,de) \end{aligned} \tag{1.4.18}$$

with

$$h(U_s(e), e) \geq 0 \quad d\mathbb{P}\otimes dt \otimes \lambda(de) \quad a.e. \ on\ \Omega \times [t,\theta] \times E. \tag{1.4.19}$$

Proof. Notice first from (1.4.6) that $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, K_s^{t,x})_{s\in[t,\theta]}$ is solution to (1.4.18)-(1.4.19). Let Y^1 be the minimal solution on $[t,\theta]$ of (1.4.18)-(1.4.19) (the existence of a minimal solution in the case of a random terminal time is obtained by similar arguments to those used in the case of a deterministic terminal time). For each $\omega\in\Omega$, there exists a minimal solution $Y^{2,\omega}$ on $[\theta(\omega),T]$ to (1.2.10)-(1.2.11) with $X = \{X_s^{\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega)}, \theta(\omega) \leq s \leq T\}$. We then have from the definition of v that $Y_{\theta(\omega)}^{2,\omega} = v\left(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega)\right)$ for all $\omega\in\Omega$. By a measurable selection result (see e.g. Thm 82 in the appendix to Ch. III in [25]), there exists $Y^2 \in \mathcal{S}^\mathbf{2}$ such that $\mathbb{P}(d\omega)$ a.s., we have $Y_{\theta(\omega)}^2(\omega) = Y_{\theta(\omega)}^{2,\omega} = v\left(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega)\right)$ and $Y_s^2(\omega) = Y_s^{2,\omega}(\omega)$ for $s\in[\theta(\omega),T]$. We then define the process $\tilde{Y}$ by $\tilde{Y}|_{[t,\theta]} = Y^1$ and $\tilde{Y}|_{(\theta,T]} = Y^2$. Hence, $\tilde{Y}$ is a solution on $[t,T]$ to (1.2.10)-(1.2.11), which implies $\tilde{Y} \geq Y^{t,x}$. Moreover, since $Y_\theta^{t,x} = v(\theta, X_\theta^{t,x})$, it follows that $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, K_s^{t,x})_{s\in[t,\theta]}$ is a solution on $[t,\theta]$ to (1.4.18)-(1.4.19). Hence $Y^1 \leq Y^{t,x}$ on $[t,\theta]$, and therefore $Y^1 = Y^{t,x}$ on $[t,\theta]$. □

Proof of Theorem 1.4.2 (i) We first prove the supersolution property of v_* to (1.4.3). Let $x\in\mathbb{R}^d$, and $(p,q,M)\in \bar{J}^- v_*(T,x)$. By same arguments as in (1.4.13), we have

$$\inf_{e\in E} h(\mathcal{H}^e[T,x,q,v_*] - v_*(T,x), e) \quad \geq \quad 0. \tag{1.4.20}$$

Moreover, since the sequence of continuous functions $(v_n)_n$ is nondecreasing and $v_n(T,.) = g$, we deduce that $v_*(T,.) \geq g$, which combined with (1.4.20), proves the viscosity supersolution property for v_* to (1.4.3).

(ii) We next prove the subsolution property of v^* to (1.4.3). We argue by contradiction and assume that there exist $x_0\in\mathbb{R}^n$, $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ such that

$$0 \;=\; (v^* - \varphi)(T,x_0) \quad = \quad \max_{[0,T]\times\mathbb{R}^d} (v^* - \varphi) \tag{1.4.21}$$

and

$$\begin{aligned}&\min\Big[\varphi(T,x_0)-g(x_0)\ ,\\ &\inf_{e\in E} h(\mathcal{H}^e[T,x_0,D_x\varphi(T,x_0),v^*]-\varphi(T,x_0),e)\Big] \quad =: \quad 2\varepsilon \ > \ 0.\end{aligned}$$

By the upper semicontinuity of v^*, the continuity of φ and its derivative, and the nonincreasing property of h, there exists an open neighbohood $\mathcal{O}$ of (T,x_0) in $[0,T]\times\mathbb{R}^d$, and $A,r>0$ such that for all $(t,x,\alpha,\beta)\in\mathcal{O}\times(-A,A)\times B(0,r)$, we have

$$\begin{aligned}\varepsilon \ \ \le \ \ &\min\Big[\varphi(t,x)-\alpha-g(x)\ ,\\ &\inf_{e\in E} h(v^*(t,x+\gamma(x,e))\\ &+c(x,\varphi(t,x)-\alpha,\sigma^\intercal(x)[D_x\varphi(t,x)+\beta])-[\varphi(t,x)-\alpha],e)\Big].\end{aligned} \tag{1.4.22}$$

Let $(t_k,x_k)_k$ be a sequence in $[0,T)\times\mathbb{R}^d$ such that

$$(t_k,x_k)\to(T,x_0) \quad \text{ and } \quad v(t_k,x_k)\to v^*(T,x_0). \tag{1.4.23}$$

Fix then $\delta>0$ such that for k large enough: $[t_k,T]\times B(x_k,\delta)\subset\mathcal{O}$, and let us define the functions φ_k by

$$\varphi_k(t,x) \quad = \quad \varphi(t,x)+\zeta\frac{|x-x_k|^2}{\delta^2}+C_k\phi\left(\frac{x-x_k}{\delta}\right)+\sqrt{T-t},$$

where $0<\zeta<A\wedge\delta r$, $\phi\in C^2(\mathbb{R}^d)$ satisfies $\phi|_{\bar B(0,1)}\equiv 0,\ \phi|_{\bar B(0,1)^c}>0$ and $\lim_{|x|\to\infty}\frac{\phi(x)}{1+|x|}=\infty$, and $C_k>0$ is a constant to be chosen below. By (1.4.21), we notice that

$$(v^*-\varphi_k)(t,x) \quad \le \quad -\zeta \qquad \text{for } \ (t,x)\in[t_k,T]\times\partial B(x_k,\delta),$$

and from the conditions on ϕ, we can choose C_k (large enough) so that

$$(v^*-\varphi_k)(t,x) \quad \le \quad -\frac{\zeta}{2} \qquad \text{for } (t,x)\in[t_k,T]\times B(x_k,\delta)^c. \tag{1.4.24}$$

Since $\frac{\partial}{\partial t}(\sqrt{T-t})\to-\infty$ as $t\nearrow T$, we have for k large enough :

$$\begin{aligned}&-\frac{\partial\varphi_k}{\partial t}-\mathcal{L}\varphi_k(t,x)-f(x,\varphi_k(t,x)-\alpha,\sigma^\intercal(x)D_x\varphi_k(t,x))\\ \ge\ \ &0 \quad \text{ for } \ (t,x,\alpha)\in[t_k,T)\times B(x_k,\delta)\times(-A+\zeta,A).\end{aligned} \tag{1.4.25}$$

Fix now $\alpha^*\in(0,A\wedge\frac{\zeta}{2}\wedge\varepsilon)$, and let us denote $\tau_k=\inf\big\{s\ge t_k\ ;\ X^k_s\neq X^k_{s^-}\big\}$, $\theta_k=\inf\big\{s\ge t_k\ ;\ X^k_s\notin B(x_k,\delta)\big\}\wedge\tau_k\wedge T$ where $X^k=X^{t_k,x_k}$. Let us then define the quadruples (Y^k,Z^k,U^k,K^k) on $[t_k,\theta_k]$ by :

$$\begin{aligned}Y^k_s \quad &= \quad \Big[\varphi_k(s,X^k_s)-\alpha^*\Big]\mathbf{1}_{\{s\in[t_k,\theta_k)\}}+v(\theta_k,X^k_{\theta_k})\mathbf{1}_{\{s=\theta_k\}}\ ,\\ Z^k_s \quad &= \quad \sigma^\intercal(X^k_{s^-})D_x\varphi_k(s,X^k_{s^-})\ ,\\ U^k_s(e) \quad &= \quad v^*(s,X^k_{s^-}+\gamma(X^k_{s^-},e))\\ &\qquad +c(X^k_{s^-},\varphi_k(s,X^k_{s^-})-\alpha^*,\sigma^\intercal(X^k_{s^-})D_x\varphi_k(s,X^k_{s^-}))\\ &\qquad -[\varphi_k(s,X^k_{s^-})-\alpha^*],\end{aligned}$$

and

$$
\begin{aligned}
K_s^k &= -\int_{t_k}^s \Big\{ \frac{\partial \varphi_k}{\partial t}(r, X_r^k) + \mathcal{L}\varphi_k(r, X_r^k) \\
&\quad + f(X_r^k, \varphi_k(r, X_r^k) - \alpha^*, \sigma^\intercal(X_r^k) D_x\varphi_k(r, X_r^k)) \Big\} dr \\
&\quad - \int_{t_k}^s \int_E (\varphi_k - \alpha^* - v^*)(r, X_{r^-}^k + \gamma(X_{r^-}^k, e)) \mu(dr, de) \\
&\quad + \Big(\varphi_k(\theta_k, X_{\theta_k}^k) - \alpha^* - v(\theta_k, X_{\theta_k}^k) \Big) \mathbf{1}_{\{s=\theta_k\}}.
\end{aligned}
$$

By construction and from Itô's formula on $\varphi_k(s, X_s^k)$, we see that (Y^k, Z^k, U^k, K^k) satisfies (1.4.18) on $[t_k, \theta_k]$. From (1.4.22), it is clear that the process U^k satisfies the constraint :

$$
h(U_t^k(e), e) \;\geq\; 0, \quad d\mathbb{P} \otimes dt \otimes \lambda(de) \;\; a.e. \;\text{ on } \Omega \times [t_k, \theta_k] \times E.
$$

Observe also that

$$
\varphi_k(\theta_k, X_{\theta_k}^k) - \alpha^* \;\geq\; v(\theta_k, X_{\theta_k}^k) \tag{1.4.26}
$$

Indeed, we have two cases:

- $(\theta_k, X_{\theta_k}^k) \in [t_k, T] \times B(x_k, \delta)^c$: since $\alpha^* < \frac{\zeta}{2}$, we have by (1.4.24),

$$
\varphi_k(\theta_k, X_{\theta_k}^k) - \alpha^* \;\geq\; v^*(\theta_k, X_{\theta_k}^k) \;\geq\; v(\theta_k, X_{\theta_k}^k).
$$

- $(\theta_k, X_{\theta_k}^k) \in [t_k, T] \times B(x_k, \delta) \subset \mathcal{O}$: since $\alpha^* \leq \varepsilon$, we have by (1.4.22)

$$
\varphi_k(\theta_k, X_{\theta_k}^k) - \alpha^* \;\geq\; \varphi(\theta_k, X_{\theta_k}^k) - \varepsilon \geq g(X_T^k) \;=\; v(\theta_k, X_{\theta_k}^k).
$$

Let us then check that K^k is nondecreasing on $[t_k, \theta_k]$. First, on $[t_k, \theta_k)$, we notice that K^k consists only in the Lebesgue term dr, and so is nondecreasing by (1.4.25). Moreover, we see that $K_{\theta_k}^k \geq K_{\theta_k^-}^k$. Indeed, there are two possible cases:

- $\theta_k < \tau_k$: then $K_{\theta_k}^k = K_{\theta_k^-}^k + \varphi_k(\theta_k, X_{\theta_k}^k) - \alpha^* - v(\theta_k, X_{\theta_k}^k)$, and by (1.4.26), we have $K_{\theta_k}^k \geq K_{\theta_k^-}^k$.

- $\theta_k = \tau_k$: then $K_{\theta_k}^k = K_{\theta_k^-}^k - (\varphi_k(\theta_k, X_{\theta_k}^k) - \alpha^* - v^*(\theta_k, X_{\theta_k}^k)) + (\varphi_k(\theta_k, X_{\theta_k}^k) - \alpha^* - v(\theta_k, X_{\theta_k}^k))$, and so $K_{\theta_k}^k \geq K_{\theta_k^-}^k$.

Therefore, the quadruple (Y^k, Z^k, U^k, K^k) is a solution on $[t_k, \theta_k]$ to (1.4.18)-(1.4.19), and by Lemma 1.4.1, we deduce that for all k,

$$
\varphi_k(t_k, x_k) - \alpha^* \;=\; \varphi(t_k, x_k) + \sqrt{T - t_k} - \alpha^* \;\geq\; v(t_k, x_k).
$$

We finally obtain a contradiction by sending k to ∞. $\square$

1.4.2 Uniqueness result

This paragraph is devoted to a uniqueness result for the QVI (1.4.1)-(1.4.3). We need to impose some additional assumptions.

(H3) There exists a nonnegative function $\Lambda \in \mathcal{C}^2(\mathbb{R}^d)$ and a positive constant ρ satisfying

(i) $\mathcal{L}\Lambda + f(., \Lambda, \sigma^{\intercal} D\Lambda) \leq \rho\Lambda$,

(ii) $\inf_{e\in E} h(\mathcal{H}^e\Lambda(x) - \Lambda(x), e) > 0$ for all $x \in \mathbb{R}^d$,

(iii) $\Lambda(x) \geq g(x)$ for all $x \in \mathbb{R}^d$,

(iv) $\lim_{|x|\to\infty} \frac{\Lambda(x)}{1+|x|} = \infty$.

Assumption **(H3)** is similar to the one made in [78] or [10], and essentially ensures the existence of a suitable strict supersolution to (1.4.1). We shall give in paragraph 1.5 some sufficient conditions for **(H3)**. This strict supersolution allows to control the nonlocal term in QVI (1.4.1)-(1.4.3) via some convex small perturbation. Thus, to deal with the dependence of f, c on y, z, we also require some convexity conditions.

(H4)

(i) The function $f(x,.,.)$ is convex in $(y,z) \in \mathbb{R} \times \mathbb{R}^d$ for all $x \in \mathbb{R}^d$.

(ii) The function $h(.,e)$ is concave in $u \in \mathbb{R}$ a for all $e \in E$.

(iii) The function $c(x,.,.,e)$ is convex in $(y,z) \in \mathbb{R} \times \mathbb{R}^d$ for all $(x,e) \in \mathbb{R}^d \times E$.

(iv) The function $c(x,.,z,e)$ is decreasing in $y \in \mathbb{R}$ for all $(x,z,e) \in \mathbb{R}^d \times \mathbb{R}^d \times E$.

Theorem 1.4.3 *Assume that* **(H3)** *and* **(H4)** *hold, and let U (resp. V) be a lsc (resp. usc) viscosity supersolution (resp. subsolution) to* (1.4.1)-(1.4.3) *satisfying a linear growth condition :*

$$\sup_{x\in\mathbb{R}^d} \frac{|U(t,x)| + |V(t,x)|}{1+|x|} \quad < \quad \infty, \quad \forall t \in [0,T].$$

Then, $U \geq V$ on $[0,T]\times\mathbb{R}^d$. Consequently, under **(H2')** *(or* **(H1')** *in the case : $h(u,e) = -u$),* **(H3)**, **(H4)**, *and* **(HE)**, *the function v in* (1.4.4) *is the unique viscosity solution to* (1.4.1)-(1.4.3) *satisfying a linear growth condition, and v is continuous on* $[0,T)\times\mathbb{R}^d$.

Proof. • *Comparison principle.* As usual, we shall argue by contradiction by assuming that

$$\sup_{[0,T]\times\mathbb{R}^d} (V-U) > 0. \tag{1.4.27}$$

1 For some $\lambda > 0$ to be chosen below, let

$$\tilde U(t,x) = e^{(\rho+\lambda)t}U(t,x)\ ,\ \tilde V(t,x) = e^{(\rho+\lambda)t}V(t,x) \ \text{ and } \ \tilde\Lambda(t,x) = e^{(\rho+\lambda)t}\Lambda(x).$$

A straightforward derivation shows that $\tilde U$ (resp. $\tilde V$) is a viscosity supersolution (resp. subsolution) to

$$\begin{aligned} \min\Big[\rho w - \frac{\partial w}{\partial t} - \mathcal{L}w - \tilde f\left(\cdot, w, \sigma^{\intercal} D_x w\right), \qquad\qquad & \qquad (1.4.28)\\ \inf_{e\in E} \tilde h\left(\cdot, \tilde{\mathcal{H}}^e w - w, e\right)\Big] &= 0, \ \text{ on } [0,T)\times\mathbb{R}^d \\ \min\Big[w(T^-,\cdot) - \tilde g, \qquad\qquad & \qquad (1.4.29)\\ \inf_{e\in E} \tilde h(T, \tilde{\mathcal{H}}^e w(T^-,\cdot) - w(T^-,\cdot), e)\Big] &= 0 \ \text{ on } \mathbb{R}^d \end{aligned}$$

where

$$\begin{aligned} \tilde f(t,x,r,q) &= e^{(\rho+\lambda)t} f\left(x, re^{-(\rho+\lambda)t}, qe^{-(\rho+\lambda)t}\right) - \lambda r \\ \tilde h(t,r,e) &= e^{(\rho+\lambda)t} h(e^{-(\rho+\lambda)t} r, e), \quad \tilde g(x) \ = \ e^{(\rho+\lambda)T} g(x) \end{aligned}$$

and

$$\tilde{\mathcal{H}}w(t,x) \quad = \quad w(t,x+\gamma(x,e)) + \tilde{c}\,(x, w(t,x), \sigma^{\intercal}(x)D_x w(t,x), e)$$

with

$$\tilde{c}(t,x,r,q,e) \quad = \quad e^{(\rho+\lambda)t}c(x, e^{-(\rho+\lambda)t}r, e^{-(\rho+\lambda)t}q, e)$$

for all $(t,x,r,q,e) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times E$. Since f is Lipschitz, we can choose λ large enough so that $\tilde{f}$ is nonincreasing in r. Denote $\tilde{W} = (1-\mu)\tilde{U} + \mu\tilde{\Lambda}$ with $\mu > 0$. By (1.4.27) and the growth condition **(H3)**(iv) of Λ, we have for μ small enough

$$\sup_{[0,T]\times\mathbb{R}^d} (\tilde{V} - \tilde{W}) = (\tilde{V} - \tilde{W})(t_0, x_0) > 0. \tag{1.4.30}$$

for some $(t_0, x_0) \in [0,T]\times\mathbb{R}^d$. Moreover from the viscosity supersolution property (1.4.28)-(1.4.29) of $\tilde{U}$, and the conditions **(H3)**(i), (ii), **(H4)**(i), (ii), (iii), we see that $\tilde{W}$ is a viscosity supersolution to

$$\rho w - \frac{\partial w}{\partial t} - \mathcal{L}w - \tilde{f}\,(\cdot, w, \sigma^{\intercal}D_x w) \quad \geq \quad 0, \quad \text{on } [0,T)\times\mathbb{R}^d, \tag{1.4.31}$$

$$\inf_{e\in E}\tilde{h}\left(\cdot, \tilde{\mathcal{H}}^e w - w, e\right) \quad \geq \quad \mu\tilde{q}, \quad \text{on } [0,T]\times\mathbb{R}^d, \tag{1.4.32}$$

where $\tilde{q}(t,x) = e^{(\rho+\lambda)t}\inf_{e\in E} h(\mathcal{H}^e\Lambda(x) - \Lambda(x), e)$ is positive on $[0,T]\times\mathbb{R}^d$ by **(H3)**(ii).

2 Denote for all $(t,x,y) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ and $n \geq 1$

$$\Theta_n(t,x,y) \quad = \quad \tilde{V}(t,x) - \tilde{W}(t,y) - \varphi_n(t,x,y),$$

with

$$\varphi_n(t,x,y) \;=\; n|x-y|^2 + |x-x_0|^4 + |t-t_0|^2.$$

By the growth assumption on U and V and **(H3)**(iii), for all n, there exists $(t_n, x_n, y_n) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ attaining the maximum of Θ_n on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. By standard arguments, we have :

$$(t_n, x_n, y_n) \quad \to \quad (t_0, x_0, x_0), \tag{1.4.33}$$

$$n|x_n - y_n|^2 \quad \to \quad 0, \tag{1.4.34}$$

$$\tilde{V}(t_n, x_n) - \tilde{W}(t_n, y_n) \quad \to \quad \tilde{V}(t_0, x_0) - \tilde{W}(t_0, x_0). \tag{1.4.35}$$

3 We now show that for n large enough

$$\inf_{e\in E}\tilde{h}(t_n, \tilde{\mathcal{H}}^e[t_n, x_n, D_x\varphi_n(t_n,x_n,y_n), \tilde{V}] - \tilde{V}(t_n,x_n), e) \quad > \quad 0. \tag{1.4.36}$$

On the contrary, up to a subsequence, we would have for all n,

$$\inf_{e\in E}\tilde{h}(t_n, \tilde{\mathcal{H}}^e[t_n, x_n, D_x\varphi_n(t_n,x_n,y_n), \tilde{V}] - \tilde{V}(t_n,x_n), e) \;\leq\; 0,$$

and so by uppersemicontinuity of $\tilde{V}$, compactness of E, there would exist a sequence (e_n) in E such that

$$\tilde{h}(t_n, \tilde{\mathcal{H}}^{e_n}[t_n, x_n, D_x\varphi_n(t_n, x_n, y_n), \tilde{V}] - \tilde{V}(t_n, x_n), e_n) \quad \leq \quad 0.$$

Moreover, by the viscosity supersolution property of $\tilde{W}$ to (1.4.32), we have

$$\tilde{h}(t_n, \tilde{\mathcal{H}}^{e_n}[t_n, y_n, -D_y\varphi_n(t_n, x_n, y_n), \tilde{W}] - \tilde{W}(t_n, y_n), e_n) \quad \geq \quad \mu\tilde{q}(t_n, y_n).$$

From the nonincreasing and the Lipschitz property of $h(., e)$, we deduce from the two previous inequalities that there exists a positive constant η such that

$$\begin{aligned} & \tilde{\mathcal{H}}^{e_n}[t_n, y_n, -D_y\varphi_n(t_n, x_n, y_n), \tilde{W}] - \tilde{W}(t_n, y_n) + \eta\tilde{q}(t_n, y_n) \\ \leq \quad & \tilde{\mathcal{H}}^{e_n}[t_n, x_n, D_x\varphi_n(t_n, x_n, y_n), \tilde{V}] - \tilde{V}(t_n, x_n), \end{aligned}$$

which is rewritten as

$$\begin{aligned} & \tilde{V}(t_n, x_n) - \tilde{W}(t_n, y_n) + \eta\tilde{q}(t_n, y_n) \\ \leq \quad & \tilde{V}(t_n, x_n + \gamma(x_n, e_n)) - \tilde{W}(t_n, y_n + \gamma(y_n, e_n)) + \Delta C_n \end{aligned} \tag{1.4.37}$$

where

$$\begin{aligned} \Delta C_n \quad = \quad & \tilde{c}\Big(t_n, x_n, \tilde{V}(t_n, x_n), \sigma^\intercal(x_n)D_x\varphi_n(t_n, x_n, y_n), e_n\Big) \\ & - \tilde{c}\Big(t_n, y_n, \tilde{W}(t_n, y_n), -\sigma^\intercal(y_n)D_y\varphi_n(t_n, x_n, y_n)\Big). \end{aligned}$$

Now, we write $\Delta C_n = \Delta C_n^1 + \Delta C_n^2 + \Delta C_n^3$, with

$$\begin{aligned} \Delta C_n^1 \quad = \quad & \tilde{c}\Big(t_n, x_n, \tilde{V}(t_n, x_n), \sigma^\intercal(x_n)D_x\varphi_n(t_n, x_n, y_n), e_n\Big) \\ & - \tilde{c}\Big(t_n, x_n, \tilde{W}(t_n, y_n), \sigma^\intercal(x_n)D_x\varphi_n(t_n, x_n, y_n), e_n\Big), \\ \Delta C_n^2 \quad = \quad & \tilde{c}\Big(t_n, x_n, \tilde{W}(t_n, y_n), \sigma^\intercal(x_n)D_x\varphi_n(t_n, x_n, y_n), e_n\Big) \\ & - \tilde{c}\Big(t_n, x_n, \tilde{W}(t_n, y_n), -\sigma^\intercal(y_n)D_y\varphi_n(t_n, x_n, y_n), e_n\Big), \\ \Delta C_n^3 \quad = \quad & \tilde{c}\Big(t_n, x_n, \tilde{W}(t_n, y_n), -\sigma^\intercal(y_n)D_y\varphi_n(t_n, x_n, y_n), e_n\Big) \\ & - \tilde{c}\Big(t_n, y_n, \tilde{W}(t_n, y_n), -\sigma^\intercal(y_n)D_y\varphi_n(t_n, x_n, y_n), e_n\Big). \end{aligned}$$

We have $\tilde{V}(t_n, x_n) - \tilde{W}(t_n, y_n) \to (\tilde{V} - \tilde{W})(t_0, x_0) > 0$ by (1.4.30) and (1.4.35). Hence, for n large enough, $\tilde{V}(t_n, x_n) \geq \tilde{W}(t_n, y_n)$, and so from the nonincreasing condition **(H4)**(iv) of c, we have $\Delta C_n^1 \leq 0$. Since $\sigma^\intercal(x_n)D_x\varphi_n(t_n, x_n, y_n) + \sigma^\intercal(y_n)D_y\varphi_n(t_n, x_n, y_n) \to 0$ by the Lipschitz condition on σ and (1.4.34), we deduce with the Lipschitz condition on c that $\limsup_{n\to\infty} \Delta C_n^2 \leq 0$. By (1.4.33) and continuity of c, we have $\lim_{n\to\infty} \Delta C_n^3 = 0$. Therefore, we obtain

$$\limsup_{n\to\infty} \Delta C_n \quad \leq \quad 0.$$

Up to a subsequence, we may assume that (e_n) converges to e_0 in E. Hence, by sending n to infinity into (1.4.37), it follows with (1.4.35) and the upper (resp. lower)-semicontinuity of $\tilde{V}$ (resp. $\tilde{W}$) that :

$$\begin{aligned}(\tilde{V}-\tilde{W})(t_0, x_0+\gamma(x_0,e_0), x_0+\gamma(x_0,e_0)) &\geq (\tilde{V}-\tilde{W})(t_0,x_0)+\eta\tilde{q}(t_0,x_0)\\ &> (\tilde{V}-\tilde{W})(t_0,x_0),\end{aligned}$$

a contradiction with (1.4.30).

4 Let us check that, up to a subsequence, $t_n < T$ for all n. On the contrary, $t_n = t_0 = T$ for n large enough, and from (1.4.36), and the viscosity subsolution property of $\tilde{V}$ to (1.4.29), we would get

$$\tilde{V}(T,x_n) \leq \tilde{g}(x_n).$$

On the other hand, by the viscosity supersolution property of $\tilde{U}$ to (1.4.29) and **(H3)**(iii), we have $\tilde{W}(T,y_n) \geq \tilde{g}(y_n)$, and so

$$\tilde{V}(T,x_n)-\tilde{W}(T,y_n) \leq \tilde{g}(x_n)-\tilde{g}(y_n).$$

By sending n to infinity, and from continuity of $\tilde{g}$, this would imply $(\tilde{V}-\tilde{W})(t_0,x_0) \leq 0$, a contradiction with (1.4.30).

5 We may then apply Ishii's lemma (see Theorem 6.1 in [34]) to $(t_n,x_n,y_n) \in [0,T)\times\mathbb{R}^d\times\mathbb{R}^d$ that attains the maximum of Θ_n, for all $n \geq 1$: there exist $(p^n_{\tilde{V}}, q^n_{\tilde{V}}, M_n) \in \bar{J}^{2,+}\tilde{V}(t_n,x_n)$ and $(p^n_{\tilde{W}}, q^n_{\tilde{W}}, N_n) \in \bar{J}^{2,-}\tilde{W}(t_n,y_n)$ such that

$$\begin{aligned}p^n_{\tilde{V}} - p^n_{\tilde{W}} &= \partial_t\varphi_n(t_n,x_n,y_n) = 2(t_n-t_0),\\ q^n_{\tilde{V}} &= D_x\varphi_n(t_n,x_n,y_n), \qquad q^n_{\tilde{W}} = -D_y\varphi_n(t_n,x_n,y_n),\end{aligned}$$

and

$$\begin{pmatrix} M_n & 0\\ 0 & -N_n\end{pmatrix} \leq A_n + \frac{1}{2n}A_n^2, \tag{1.4.38}$$

where $A_n = D^2_{(x,y)}\varphi_n(t_n,x_n,y_n)$. From the viscosity supersolution property of $\tilde{W}$ to (1.4.31), we have

$$\begin{aligned}\rho\tilde{W}(t_n,y_n) - p^n_{\tilde{W}} + \langle b(y_n), D_y\varphi(t_n,x_n,y_n)\rangle - \frac{1}{2}\mathrm{tr}(\sigma(y_n)\sigma^{\intercal}(y_n)N_n)&\\ -\tilde{f}(t_n,y_n,\tilde{W}(t_n,y_n),-\sigma^{\intercal}(y_n)D_y\varphi(t_n,x_n,y_n)) &\geq 0.\end{aligned}$$

On the other hand, from (1.4.36) and the viscosity subsolution property of $\tilde{V}$ to (1.4.28), we have

$$\begin{aligned}\rho\tilde{V}(t_n,x_n) - p^n_{\tilde{V}} - \langle b(x_n), D_x\varphi(t_n,x_n,y_n)\rangle - \frac{1}{2}\mathrm{tr}(\sigma(x_n)\sigma^{\intercal}(x_n)M_n)&\\ -\tilde{f}(t_n,x_n,\tilde{V}(t_n,x_n),\sigma^{\intercal}(x_n)D_x\varphi(t_n,x_n,y_n)) &\leq 0.\end{aligned}$$

By subtracting the two previous inequalities, we obtain

$$\begin{aligned}\rho(\tilde{V}(t_n,x_n)-\tilde{W}(t_n,y_n)) &\leq p^n_{\tilde{V}}-p^n_{\tilde{W}}+\Delta F_n \\ &\quad+\langle b(x_n),D_x\varphi_n(t_n,x_n,y_n)\rangle+\langle b(y_n),D_y\varphi_n(t_n,x_n,y_n)\rangle \\ &\quad+\frac{1}{2}\mathrm{tr}\left(\sigma(x_n)\sigma^{\intercal}(x_n)M_n-\sigma(y_n)\sigma^{\intercal}(y_n)N_n\right), \qquad (1.4.39)\end{aligned}$$

where

$$\begin{aligned}\Delta F_n &= \tilde{f}(t_n,x_n,\tilde{V}(t_n,x_n),\sigma^{\intercal}(x_n)D_x\varphi_n(t_n,x_n,y_n)) \\ &\quad-\tilde{f}(t_n,y_n,\tilde{W}(t_n,y_n),-\sigma^{\intercal}(y_n)D_y\varphi_n(t_n,x_n,y_n)).\end{aligned}$$

From (1.4.33), we have $p^n_{\tilde{V}}-p^n_{\tilde{W}}\to 0$ as n goes to infinity. From the Lipschitz property of b, and (1.4.34), we have

$$\lim_{n\to\infty}\Big(\langle b(x_n),D_x\varphi_n(t_n,x_n,y_n)\rangle+\langle b(y_n),D_y\varphi_n(t_n,x_n,y_n)\rangle\Big) = 0.$$

As usual, from (1.4.38), (1.4.33), (1.4.34), and the Lipschitz property of σ, we have

$$\limsup_{n\to\infty}\mathrm{tr}\left(\sigma(x_n)\sigma^{\intercal}(x_n)M_n-\sigma(y_n)\sigma^{\intercal}(y_n)N_n\right) \leq 0.$$

Moreover, by the same arguments as for $\tilde{c}$, using the nonincreasing property of $\tilde{f}$ in its third variable, and the Lipschitz property of $\tilde{f}$, we have

$$\limsup_{n\to\infty}\Delta F_n \leq 0.$$

Therefore, by sending $n\to\infty$ into (1.4.39), we conclude with (1.4.35) that $\rho(\tilde{V}-\tilde{W})(t_0,x_0)$ ≤ 0, a contradiction with (1.4.30).

• *Uniqueness for v.* The uniqueness result is then a direct consequence of the comparison principle, and the continuity of v on $[0,T)\times\mathbb{R}^d$ follows from the fact that in this case $v_*=v^*$. □

Remark 1.4.3 As a byproduct of the comparison principle in Theorem 1.4.3, we get the continuity of the value function v on $[0,T)\times\mathbb{R}^d$. Since the jump-diffusion process X is quasi-left continuous, then so is the minimal solution $Y_t=v(t,X_t)$ to the BSDE with constrained jumps, and the penalized approximation $Y^n_t=v_n(t,X_t)$. This implies that the predictable projections ${}^pY$ and ${}^pY^n$, respectively of Y and Y^n, are equal to ${}^pY_t=Y_{t^-}$ and ${}^pY^n_t=Y^n_{t^-}$. Therefore, $Y_{t^-}=\lim_{n\to\infty}Y^n_{t^-}$. From the weak version of Dini's theorem, see [26] p. 202, this yields the uniform convergence of Y^n on $[0,T]$, i.e. $\lim_{n\to\infty}\sup_{t\in[0,T]}|Y^n_t-Y_t|=0$, and so by the dominated convergence theorem, the convergence of Y^n to Y in $\mathcal{S}^{\mathbf{2}}$:

$$\lim_{n\to\infty}\|Y^n-Y\|_{\mathcal{S}^{\mathbf{2}}} = 0. \qquad (1.4.40)$$

Then, by applying Itô's formula to $t\mapsto\mathbb{E}|Y_t-Y^n_t|$ a in the proof of Theorem 1.3.1, we get from the convergence of Y^n to Y in $\mathcal{S}^{\mathbf{2}}$ that (Z^n,V^n) converges to (Z,V) in $\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})$ and that K is continuous.

1.5 Some sufficient conditions for (H2') and (H3)

In this section, we provide various explicit conditions on the coefficients model, which ensure that the general assumptions **(H2')** and **(H3)** hold true.

1.5.1 Existence of the solution to BSDE with jump constraint

We first consider a case where we have upper bounds for the coefficients and $h(u,e) = -u$.

Proposition 1.5.3 *Suppose that $h(u,e) = -u$, and assume that there exist real constants C_1, C_2 and $\eta \in \mathbb{R}^d$ such that*

$$g(x) \ \leq \ C_1 + \langle \eta, x \rangle \ , \ c(x,y,z,e) + \langle \eta, \gamma(x,e) \rangle \ \leq \ 0 \qquad (1.5.41)$$
$$\text{and} \ \ f(x,y,z) + \langle \eta, b(x) \rangle \ \leq \ C_2,$$

for all $(x,y,z,e) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times E$. Then **(H2')** *holds true.*

Proof. Let us define a quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U})$ by : $\tilde{Y}_t = C_1 + C_2(T-t) + \langle \eta, X_t \rangle$ for $t < T$, $\tilde{Y}_T = g(X_T)$, $\tilde{Z}_t = \sigma(X_{t^-}).\eta$, $\tilde{U}_t(e) = 0$ and

$$\begin{aligned}
\tilde{K}_t &= \int_0^t \Big\{ C_2 - \eta \cdot b(X_s) - f(X_s, \tilde{Y}_s, \tilde{Z}_s) \Big\} ds \\
&\quad - \int_0^t \int_E \Big\{ c(X_{s-}, \tilde{Y}_{s-}, \tilde{Z}_s, e) + \langle \eta, \gamma(X_{s-}, e) \rangle \Big\} \mu(ds,de), \quad t < T, \\
\tilde{K}_T &= \tilde{K}_{T-} + C_1 + \langle \eta, X_T \rangle - g(X_T).
\end{aligned}$$

From (1.5.41), the process $\tilde{K}$ is clearly nondecreasing. Moreover, from the dynamics of X, and by construction, we see that the quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U})$ satisfies (1.2.10)-(1.2.13) and the function $\tilde{v}(t,x) = C_1 + C_2(T-t) + \eta.x$ clearly satisfies a linear growth condition. □

We next give an example inspired by [10] where the jumps of X vanish as X goes out of a ball centered in zero in the case of impulse control.

Proposition 1.5.4 *Suppose that $h(u,e) = -u$, f, c does not depend on y, z, and assume that $c \leq 0$, $\gamma = 0$ on $\{x \in \mathbb{R}^d \ : \ |x| \geq C_1\} \times E$ for some $C_1 > 0$. Then,* **(H2')** *holds true.*

Proof. We consider the function v :

$$v(t,x) \ = \ \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \Big[g(X_T^{t,x}) + \int_t^T f(X_s^{t,x}) ds + \int_t^T \int_E c(X_{s^-}^{t,x}, e) \mu(ds,de) \Big].$$

Since $c \leq 0$, and the choice of $\nu = 1$ corresponds to the probability measure $\mathbb{P}^1 = \mathbb{P}$, we see that $\hat{v} \leq v \leq \bar{v}$ where

$$\begin{aligned}
\hat{v}(t,x) &= \mathbb{E}\Big[g(X_T^{t,x}) + \int_t^T f(X_s^{t,x}) ds + \int_t^T \int_E c(X_{s^-}^{t,x}, e) \mu(ds,de) \Big] \\
\bar{v}(t,x) &= \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \Big[g(X_T^{t,x}) + \int_t^T f(X_s^{t,x}) ds \Big].
\end{aligned}$$

The function $\hat{v}$ clearly satisfies a linear growth condition by the linear growth conditions on g, f, c and the standard estimate for X. Moreover, under the assumptions on the jump coefficient γ, it is shown in [10] that $\bar{v}$ satisfies a linear growth condition. Therefore, $\hat{v}$ also satisfies a linear growth condition.

Let us now define the process $Y_t = v(t, X_t)$, which is then equal to

$$Y_t \quad = \quad \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^\nu \Big[g(X_T) + \int_t^T f(X_s) ds + \int_t^T \int_E c(X_{s^-}, e) \mu(ds, de) \Big| \mathcal{F}_t \Big],$$

and lies in $\mathcal{S}^\mathbf{2}$ from the linear growth condition, and the estimate (1.2.2) for X. From Theorem 1.2.1, we then know that there exists $(Z, U, K) \in \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$ such that (Y, Z, U, K) is the minimal solution to (1.2.10)-(1.2.13), and so **(H2')** is satisfied. □

We finally consider a case for general constraint function h.

Proposition 1.5.5 *Assume that there exists a Lipschitz function $w \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying a linear growth condition, supersolution to* (1.4.3), *and such that*

$$\langle b, Dw \rangle + \frac{1}{2} tr(\sigma \sigma^\intercal D^2 w) + f(\cdot, w, \sigma^\intercal Dw) \quad \leq \quad C, \qquad on \ \mathbb{R}^d,$$

for some constant C. Then **(H2')** *holds true.*

Proof. Let us define a quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ by

$$\tilde{Y}_t \quad = \quad w(X_t) + C(T - t), \quad t < T, \qquad \tilde{Y}_T = g(X_T),$$

$\tilde{Z}_t = \sigma^\intercal(X_{t^-}) Dw(X_{t^-})$, $\tilde{U}_t(e) = w(X_{t^-} + \gamma(X_{t^-}, e)) + c(X_{t^-}, \tilde{Y}_{t^-}, \tilde{Z}_t, e) - w(X_{t^-})$, and

$$\begin{aligned}
\tilde{K}_t \quad &= \quad \int_0^t [C - \langle b(X_s), Dw(X_s) \rangle \\
&\qquad - \frac{1}{2} \mathrm{tr}\{\sigma(X_s) \sigma^\intercal(X_s) D^2 w(X_s)\} - f(X_s, \tilde{Y}_s, \tilde{Z}_s)] ds, \ \ t < T, \\
\tilde{K}_T \quad &= \quad \tilde{K}_{T^-} + w(X_T) - g(X_T).
\end{aligned}$$

From the conditions on w, we see that $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U})$ lies in $\mathcal{S}^\mathbf{2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2}$. Moreover, by Itô's formula to $w(X_t)$ and the supersolution property of w to (1.4.3), we conclude that $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U})$ is solution to (1.2.10)-(1.2.11), and $\tilde{v}(t, x) = w(t, x) + C(T - t)$ satisfies a linear growth condition. □

1.5.2 The strict supersolution condition (H3)

We give a sufficient condition for **(H3)** in the usual case where f and c do not depend neither on y nor on z.

Proposition 1.5.6 *Consider the case where h is given by*

$$h(u,e) = -u.$$

Assume that there exists a constant $\alpha > 0$ such that

$$\begin{aligned} -\alpha &< |x+\gamma(x,e)|^2 - |x|^2 \quad \forall (x,e) \in \mathbb{R}^d \times E \\ \beta &:= \inf_{(x,e)\in\mathbb{R}^d\times E} \frac{-c(x,e)}{|x+\gamma(x,e)|^2 - |x|^2 + \alpha} > 0 \end{aligned}$$

Then assumption **(H3)** *holds true.*

Proof. We set $\Lambda(x) := \beta|x|^2 + \zeta$ with ζ large enough so that $\Lambda \geq g$, i.e. **(H3)**(iii) is satisfied. A straightforward computation shows that

$$\inf_{e\in E} h(\mathcal{H}^e\Lambda(x) - \Lambda(x), e) \geq \alpha\beta > 0$$

and hence **(H3)** (ii) is satisfied. Clearly, **(H3)** (iv) holds as well. Finally, it follows from the linear growth assumption on b and σ that **(H3)** (i) holds for a sufficiently large parameter ρ. □

Chapter 2

Constrained BSDEs with jumps : Application to optimal switching

Abstract : This paper enlarges the class of backward stochastic differential equation (BSDE) with jumps, adding some general constraints on all the components of the solution. Via a penalization procedure, we provide an existence and uniqueness result for these so-called constrained BSDEs with jumps. This new type of BSDE offers a nice and practical unifying framework to represent and generalize the notions of constrained BSDEs studied in [65], BSDEs with constrained jumps introduced recently by [46], as well as multidimensional BSDEs with oblique reflection presented in [44] and [41]. For example, a switching problem, represented by a multidimensional BSDE with oblique reflection, see [44], can be directly solved through a one dimensional constrained BSDE with jumps. This result is very promising from a numerical point of view for the resolution of high dimensional switching problems. All the arguments presented here rely on probabilistic tools. This allows in particular to represent non-Markovian switching problems, where, for example, the switching regime influences the dynamics of the underlined diffusion.

Keywords: Switching problems, BSDE with jumps, Constrained BSDE, Reflected BSDE.

2.1 Introduction

Since its introduction by Pardoux and Peng in [61], the notion of Backward Stochastic Differential Equations (BSDEs in short) has been widely extended. In particular, it appeared as a very powerful tool to solve partial differential equations (PDE) and corresponding stochastic optimization problems. Several generalizations of this notion are based on the addition of new constraints on the solution. First, El Karoui et al. [29] study the case where the component Y is forced to stay above a given process, leading to the notion of reflected BSDEs related to optimal stopping and obstacle problems. Motivated by super replication problems under portfolio constraints, Cvitanic et al. [24] consider the case where the component Z is constrained to stay in a fixed convex set. More recently, Kharroubi et al. [46] introduce a constraint on the jump component U of the BSDE, providing a representation of solutions to a class of PDE, called quasi-variational inequalities, arising in optimal impulse control problems. Generalizing the results of El Karoui et al. [29] in a multi-dimensional framework, Hu and Tang [44] followed by Hamadène and Zhang [41] consider BSDEs with oblique reflections and connect them with systems of variational inequalities and optimal switching problems. Nevertheless, they only consider cases where the switching strategy does not affect the dynamics of the underlying diffusion. Our paper introduce the notion of constrained BSDEs with jumps, which offers in particular a nice and natural probabilistic representation for these types of switching problems. This new notion essentially unifies and extends the notions of constrained BSDE without jumps, BSDE with constrained jumps as well as multidimensional BSDE with oblique reflections.

Let illustrate our presentation by the example of a switching problem and introduce an underlying diffusion process, whose dynamics are given by

$$X_t^\alpha = X_0 + \int_0^t b(X_u^\alpha, \alpha_u)du + \int_0^t \sigma(X_u^\alpha, \alpha_u)dW_u, \quad 0 \le t \le T, \tag{2.1.1}$$

where α is a switching control process valued in $\{1, \dots, m\}$. We consider the following switching control problem defined by

$$\sup_\alpha \mathbb{E}\Big[g(X_T^\alpha, \alpha_T) + \int_0^T f(X_s^\alpha, \alpha_s)ds + \sum_{0<\tau_k\le T} c(\alpha_{\tau_k^-}, \alpha_{\tau_k})\Big], \tag{2.1.2}$$

where $(\tau_k)_k$ denotes the jump times of the control α. This type of stochastic control problem is typically encountered by an agent maximizing the production rentability of a given good by switching between m possible modes of production based on different commodities. A switch is penalized by a given cost function c and, since the agent is a large actor on the market, the chosen mode of production influences the dynamics of the corresponding commodities. One of the mode of production can also be interpreted as a strategy where the agent directly buys the good on a financial market. As observed by Tang and Yong [78], the value function associated to this problem interprets on $[0,T]$ as the unique viscosity solution of a given coupled system of variational inequalities. The difficulty in the derivation of a BSDE representation for this type of problem is, firstly, the dependence of the solution in mode $i \in \{1,\dots,m\}$ with respect to the global solution in all possible modes, and secondly, the dependence in the control of the drift and the volatility of X. As recently observed in [10], the unique solution to the corresponding system of variational inequalities interprets as the value function of a well suited stochastic target problem associated to a diffusion with jumps. Using entirely probabilistic arguments, the BSDE representation provided in this paper relies on this type of correspondence. In our approach, we let artificially the strategy jump randomly between the different modes of production. As in [63], this allows to retrieve in the jump component of a one-dimensional backward process, some information regarding the solution in the other modes of production. Indeed, let us introduce a pure jump process $(I_t)_{0\le t\le T}$ based on an independent random measure μ and construct the underlying process $(X_t^{I_t})_{0\le t\le T}$, whose dynamics are based on the random mode of production I according to equation (2.1.1). Let consider the following constrained BSDE associated to the two dimensional forward process (I,X^I) (called transmutation-diffusion process in [63]) and defined on $[0,T]$ by:

$$\left\{\begin{array}{rcl} Y_t & = & g(X_T^{I_T}) + \int_t^T f(X_s^{I_s}, I_s)ds + K_T - K_t \\ & & -\int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds,di)\,, \quad 0\le t\le T\,, \\ U_t(i) & \ge & c(i, I_{t-}), \qquad d\mathbb{P}\otimes dt\otimes\lambda(di) \;\; a.e. \end{array}\right. \tag{2.1.3}$$

We prove in this paper that (2.1.3) has one unique minimal solution which indeed relates directly to the solution of the corresponding switching problem (2.1.2).

In order to unify our results with the one based on BSDE with oblique reflection considered in [44] or [41], we extend this approach and introduce the notion of constrained BSDE with jumps whose solution (Y,Z,U,K) satisfies the general dynamics

$$\begin{array}{rcl} Y_t & = & \xi + \displaystyle\int_t^T f(s,Y_s,Z_s,U_s)ds + K_T - K_t \\ & & \qquad - \displaystyle\int_t^T \langle Z_s, dW_s\rangle - \int_t^T\int_{\mathcal{I}} U_s(i)\mu(ds,di)\,, \quad 0\le t\le T \end{array} \tag{2.1.4}$$

a.s., for $0\le t\le T$, as well as the constraint

$$h(t, Y_{t^-}, Z_t, U_t(i), i) \;\; \ge \;\; 0, \qquad d\mathbb{P}\otimes dt\otimes\lambda(di) \;\; a.e.\,, \tag{2.1.5}$$

where f and h are given random functions. This notion of constrained BSDEs with jumps also includes BSDEs studied by Buckdahn and Hu [16] for the pricing of american options with underlying diffusions with jumps and portfolio constraints. Through a penalization argument, we provide, in Section 2, existence and uniqueness of the minimal solution to this type of constrained BSDE with jumps (2.1.4)-(2.1.5). For this purpose, we present in the Appendix an extension of Peng's monotonic limit theorem [64] to the framework of BSDEs with jumps. In Section 2, we mainly extend and unify the existing literature in three directions:

- We generalize the notion of BSDE with constrained jumps considered in [46], letting the driver function f depend on U and considering general constraint function h depending on all the components of the solution.
- We add some jumps in the dynamics of constrained BSDE studied in [65] and let the coefficients depend on the jump component U.
- In the case of general non linear switching problems considered in [41], the minimal solution to our BSDE interprets nicely in terms of solution to their corresponding BSDE with oblique reflections.

The constrained BSDEs with jumps offer a natural unifying framework to represent these three distinct types of BSDE. We believe that the representation of a multidimensional obliquely reflected BSDEs by a one dimensional constrained BSDE with jumps is numerically very promising. It offers the possibility to generalize the results of [12] and develop an entirely probabilistic algorithm for solving Markovian switching problems. Furthermore, this algorithm could solve high dimensional systems of variational inequalities, which relates directly to multidimensional BSDEs with oblique reflections, see [44] for more details. The algorithm as well as the Feynman Kac representation of general constrained BSDEs with jumps is currently under study and will appear in [33].

Like all the arguments of the paper, the proof relating constrained BSDEs with jumps and BSDEs with oblique reflections only relies on probabilistic arguments and can be applied in a non-Markovian setting. In particular, we provide a new multidimensional comparison theorem, based on viability property for super-solutions to BSDE. Nevertheless, the class of reflected BSDE studied in [44] or [41] does not allow for the consideration of switching problems where the dynamics of the underlying diffusion depends in a general manner on the current switching regime. The last section of the paper deals with this type of general non Markovian switching problem, typically of the form of (2.1.2) where the functions g, f and c are possibly random. We relate the value process of the optimal switching problem to a well chosen family of multidimensional BSDE with oblique reflection. We finally link via a penalization procedure this family of reflected BSDEs with a member of the class of one-dimensional constrained BSDE with jumps. Therefore, constrained BSDEs with jumps offer also a nice probabilistic representation for general switching problems, even in a non-Markovian framework.

Notations. Throughout this paper we are given a finite terminal time T and a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$, and a Poisson random measure μ on $\mathbb{R}_+ \times \mathcal{I}$, where $\mathcal{I} = \{1, \ldots, m\}$, with intensity measure $\lambda(di)dt$ for some finite measure λ on $\mathcal{I}$ with $\lambda(i) > 0$ for all $i \in \mathcal{I}$. We set $\tilde{\mu}(dt, di) = \mu(dt, di) - \lambda(di)dt$ the compensated measure associated to μ. $\sigma(\mathcal{I})$ denotes the σ-algebra of subsets of $\mathcal{I}$. For $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$, we set $|x| = \sqrt{|x_1|^2 + \cdots + |x_\ell|^2}$ the euclidean norm. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ (resp.$\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$) the augmentation of the natural filtration generated by W and μ (resp. by W), and by $\mathcal{P}$ the σ-algebra of predictable subsets of $\Omega \times [0, T]$. We denote by $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathcal{S}^{\mathbf{c},\mathbf{2}}_{\mathbb{F}}$, $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$) the set of real-valued càd-làg $\mathbb{F}$-adapted (resp. continous $\mathbb{F}$-adapted, càd-làg $\mathbb{G}$-adapted) processes $Y = (Y_t)_{0\leq t\leq T}$ such that

$$\|Y\|_{\mathcal{S}^{\mathbf{2}}} \quad := \quad \left(\mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big]\right)^{\frac{1}{2}} \quad < \quad \infty.$$

where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ (resp. $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$) denotes the completed filtration generated by W (resp. by W and μ). $\mathbf{L^p(0,T)}$, $p \geq 1$, is the set of real-valued measurable processes $\phi = (\phi_t)_{0\leq t\leq T}$ such that

$$\mathbb{E}\Big[\int_0^T |\phi_t|^p dt\Big] \quad < \quad \infty,$$

and $\mathbf{L^p_{\mathbb{F}}(0,T)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(0,T)}$) is the subset of $\mathbf{L^p(0,T)}$ consisting of $\mathbb{F}$–progressively measurable (resp. $\mathbb{G}$-progressively measurable) processes.
$\mathbf{L^p_{\mathbb{F}}(W)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(W)}$), $p \geq 1$, is the set of $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable (resp. $\mathcal{P}$-measurable) processes $Z = (Z_t)_{0\leq t\leq T}$ such that

$$\|Z\|_{\mathbf{L^p(W)}} \quad := \quad \left(\mathbb{E}\Big[\int_0^T |Z_t|^p dt\Big]\right)^{\frac{1}{p}} \quad < \quad \infty.$$

$\mathbf{L^p}(\tilde{\mu})$, $p \geq 1$, is the set of $\mathcal{P} \otimes \mathcal{E}$-measurable maps $U : \Omega \times [0, T] \times \mathcal{I} \to \mathbb{R}$ such that

$$\|U\|_{\mathbf{L^p}(\tilde{\mu})} \quad := \quad \left(\mathbb{E}\Big[\int_0^T \int_{\mathcal{I}} |U_t(i)|^p \lambda(di)dt\Big]\right)^{\frac{1}{p}} \quad < \quad \infty.$$

$\mathbf{A}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$) is the closed subset of $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$) consisting of nondecreasing processes $K = (K_t)_{0\leq t\leq T}$ with $K_0 = 0$.

For ease of notation, we omit in all the paper the dependence in $\omega \in \Omega$, whenever it is explicit.

2.2 Constrained Backward SDEs with jumps

This section is devoted to the presentation of constrained Backward SDEs with jumps in a framework generalizing the one considered in [46] and [65]. Namely we allow the driver function to depend on the jump component of the backward process and we extend the class of possible constraint functions by letting them depend on all the components of the solution to the BSDE. We provide here an existence and uniqueness result for this type of BSDEs and remark that they are closely related to the notion of BSDEs with oblique reflections studied by [44] and [41]. All the arguments presented here rely solely on probabilistic tools.

2.2.1 Formulation

A constrained BSDE with jumps is characterized by three objects :

- a terminal condition, i.e. a $\mathcal{G}_T$-measurable random variable ξ,
- a generator function, i.e. a progressively measurable map $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}} \to \mathbb{R}$,
- a constraint function, i.e. a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \sigma(\mathcal{I})$-measurable map $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ such that $h(\omega, t, y, z, ., i)$ is non-increasing for all $(\omega, t, y, z, i) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{I}$.

Definition 2.2.1 *A solution to the corresponding constrained BSDE with jumps is a quadruple $(Y, Z, U, K) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2_{\mathbb{G}}$ satisfying*

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + K_T - K_t \\ &\qquad - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di) \ , \quad 0 \le t \le T \ , \end{aligned} \tag{2.2.1}$$

for $0 \le t \le T$ a.s., as well as the constraint

$$h(t, Y_{t^-}, Z_t, U_t(i), i) \ \geq \ 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(di) \ \ a.e. \ . \tag{2.2.2}$$

Furthermore, (Y, Z, U, K) is referred to as the minimal solution to (2.2.1)-(2.2.2) whenever, for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ to (2.2.1)-(2.2.2), we have $Y_. \le \tilde{Y}_.$ a.s.. In this case, Y naturally interprets in the terminology of Peng [64] as the smallest supersolution to (2.2.1)-(2.2.2).

Remark 2.2.1 In the case where the driver function f does not depend on U and the constraint function h is of the form $h(u + c(t, y, z), i)$, observe that this BSDE exactly fits in the framework considered in [46]. In the Brownian case (i.e. no jump component), this type of BSDEs has been studied in [65].

In order to derive an existence and uniqueness result for solutions to this type of BSDE, we require the classical Lipschitz and linear growth conditions on the coefficients as well as a constraint on the dependence of the driver function in the jump component of the BSDE. We regroup these conditions in the following assumption.

(H0)

(i) There exists a constant $k > 0$ such that the functions f and h satisfy $\mathbb{P}-$a.s. the uniform Lipschitz property

$$\begin{aligned} |f(t,y,z,(u_j)_j) - f(t,y',z',(u'_j)_j)| &\leq k|(y,z,(u_j)_j) - (y',z',(u'_j)_j))|\,, & (2.2.3) \\ |h(t,y,z,u_i,i) - h(t,y',z',u'_i,i)| &\leq k|(y,z,u_i) - (y',z',u'_i)| & (2.2.4) \end{aligned}$$

for all $(t,i,y,z,(u_j)_j,y',z',(u'_j)_j) \in [0,T] \times \mathcal{I} \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}]^2$.

(ii) The coefficients f and h satisfy the following growth linear condition : there exists a constant C such that $\mathbb{P}-$a.s.

$$|f(t,y,z,(u_j)_j)| + |h(t,y,z,u_i,i)| \leq C(1 + |y| + |z| + |(u_j)_j|) \qquad (2.2.5)$$

for all $(t,i,y,z,(u_j)_j) \in [0,T] \times \mathcal{I} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}$.

(iii) There exist two constants $C_1 \geq C_2 > -1$ such that $\mathbb{P}-$ a.s.

$$f(t,y,z,u) - f(t,y,z,u') \leq \int_{\mathcal{I}} (u_i - u'_i)\gamma_t^{y,z,u,u'}(i)\lambda(di),$$

for all $(y,z,u,u') \in \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^{\mathcal{I}}]^2$, where $\gamma^{y,z,u,u'} : \Omega \times [0,T] \times \mathcal{I} \to \mathbb{R}$ is $\mathcal{P} \otimes \sigma(\mathcal{I})-$measurable and satisfies $C_2 \leq \gamma^{y,z,u,u'} \leq C_1$.

Remark 2.2.2 Under Assumption **(H0)** (i) and (ii), existence and uniqueness of a solution (Y,Z,U,K) to the BSDE (2.2.1) with $K = 0$ follows from classical results on BSDEs with jumps, see [5] or [77] for example. In order to add the h-constraint (2.2.2), one needs as usual to relax the dynamics of Y by adding the non decreasing process K in (2.2.1). In mathematical finance, the purpose of this new process K is to increase the super replication price Y of a contingent claim, under additional portfolio constraints. In order to find a minimal solution to the constrained BSDE (2.2.1)-(2.2.2), the nondecreasing property of h is crucial for stating comparison principles needed in the penalization approach. The simpler example of constraint function to keep in mind is $h(.,u,i) = c(.,i) - u$, i.e. upper-bounded jumps constraint.

Remark 2.2.3 Part (iii) of Assumption **(H0)** constrains the form of the dependence of the driver in the jump component of the BSDE. It is inspired from [73] and will ensure comparison results for BSDEs driven by this type of driver.

2.2.2 Existence, uniqueness and approximation by penalization

In this paragraph, we provide an existence and uniqueness result for solutions to constrained BSDEs with jumps of the form (2.2.1)-(2.2.2). This result requires an extension of Peng's monotonic limit theorem to the case of BSDE with jump, which is presented in the Appendix.

The proof relies on a classical penalization argument and we introduce the following sequence of BSDEs with jumps

$$
\begin{aligned}
Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n)ds + n\int_t^T \int_{\mathcal{I}} h^-(s, Y_s^n, Z_s^n, U_s^n(i), i)\lambda(di)ds \qquad (2.2.6)\\
&\quad - \int_t^T \langle Z_s^n, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i)\mu(ds, di), \quad 0 \le t \le T,\ n \in \mathbb{N},
\end{aligned}
$$

where $h^-(t, y, z, u, i) := \max(-h(t, y, z, u, i), 0)$ is the negative part of the function h. Under Assumption **(H0)**, the Lipschitz property of the coefficients f and h ensures existence and uniqueness of a solution $(Y^n, Z^n, U^n) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ to (2.2.6), see [5] or [77] . Let first state two comparison results ensuring a monotonic convergence of the sequence $(Y^n)_n$ under the additional assumption

(H1) There exists a quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ solution of (2.2.1)-(2.2.2).

This assumption may appear restrictive but is classical and we present in Section 2.2.3 some examples where **(H1)** is satisfied, see Remark 2.2.4 below for more details. Let us first state a general comparison theorem for BSDEs with jumps.

Lemma 2.2.1 *Let $f_1, f_2 : \Omega\times[0, T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{\mathcal{I}} \to \mathbb{R}$ two generators satisfying Assumption **(H0)** and $\xi_1, \xi_2 \in \mathbf{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$. Let $(Y^1, Z^1, U^1) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ satisfying*

$$
Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1, U_s^1)ds - \int_t^T \langle Z_s^1, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s^1(i)\mu(ds, di)\ ,
$$

for $0 \le t \le T$ a.s., and $(Y^2, Z^2, U^2, K^2) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying

$$
Y_t^2 = \xi^2 + \int_t^T f_2(s, Y_s^2, Z_s^2, U_s^2)ds - \int_t^T \langle Z_s^2, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s^2(i)\mu(ds, di) + K_T^2 - K_t^2\ ,
$$

for $0 \le t \le T$ a.s.. If $\xi^1 \le \xi^2$ and $f_1(t, Y_t^1, Z_t^1, U_t^1) \le f_2(t, Y_t^1, Z_t^1, U_t^1)$ for all $t \in [0, T]$ then we have $Y_t^1 \le Y_t^2$ for all $t \in [0, T]$.

Proof. Let us denote $\bar{Y} := Y^2 - Y^1$, $\bar{Z} := Z^2 - Z^1$, $\bar{U} := U^2 - U^1$, $\bar{f} = f_2(., Y^2, Z^2, U^2) - f_1(., Y^1, Z^1, U^1)$ and $\bar{\xi} = \xi^2 - \xi^1$ so that

$$
\begin{aligned}
\bar{Y}_t &= \bar{\xi} + \int_t^T \bar{f}_s ds - \int_t^T \langle \bar{Z}_s, dW_s\rangle\\
&\quad - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i)\mu(ds, de) + \tilde{K}_T - \tilde{K}_t, \quad 0 \le t \le T,\ a.s. \qquad (2.2.7)
\end{aligned}
$$

Let now define the process a by

$$
a_t = \frac{f_2(t, Y_t^2, Z_t^2, U_t^2) - f_2(t, Y_t^1, Z_t^2, U_t^2)}{\bar{Y}_t}\mathbf{1}_{\{\bar{Y}_t\neq 0\}}\ ,
$$

and b the $\mathbb{R}^d$-valued process defined component by component by

$$b_t^k = \frac{f_2(t, Y_t^1, Z_t^{(k-1)}, U_t^2) - f_2(t, Y_t^1, Z_t^{(k)}, U_t^2)}{V_t^k} \mathbf{1}_{\{V_t^k \neq 0\}}, \quad k = 1, \ldots, d,$$

where $Z_t^{(k)}$ is the $\mathbb{R}^d$-valued random vector whose k first components are those of Z^1 and whose $(d-k)$ lasts are those of Z^2, and V_t^k is the k-th component of $Z_t^{(k-1)} - Z_t^{(k)}$.

Notice that the processes a, b are bounded since f is Lipschitz continuous. Observe also that the process $\hat{K}$ defined by

$$\hat{K}_t = K_t^2 - \int_0^t \int_{\mathcal{I}} \gamma_s^{Y_{s^-}^1, Z_s^1, U_s^2, U_s^1} \bar{U}_s(i)\lambda(di)ds + \int_0^t (f_2(s, Y_s^1, Z_s^1, U_s^2) - f_1(s, Y_s^1, Z_s^1, U_s^1))ds$$

is an increasing process according to **(H0)** (iii) and $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$ for all $t \in [0, T]$. With these notations, we rewrite (2.2.7) as:

$$\begin{aligned}\bar{Y}_t &= \bar{\xi} + \int_t^T \left(a_s \bar{Y}_s + b_s.\bar{Z}_s + \int_{\mathcal{I}} \gamma_s^{Y_{s^-}^1, Z_s^1, U_s^2, U_s^1} \bar{U}_s(i)\lambda(di) \right) ds \\ &\qquad - \int_t^T \langle \bar{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i)\mu(ds, de) + \hat{K}_T - \hat{K}_t.\end{aligned}$$

Consider now the positive process Γ solution to the s.d.e.:

$$d\Gamma_t = \Gamma_{t^-} \left(a_t dt + \langle b_t, dW_t \rangle + \int_{\mathcal{I}} \gamma_s^{Y_{s^-}^1, Z_s^1, U_s^2, U_s^1} \mu(di, ds) \right), \quad \Gamma_0 = 1.$$

Notice that Γ lies in $\mathcal{S}_{\mathbb{G}}^2$ since a, b and γ are bounded, and Γ is positive according to **(H3)** (iii). A direct application of Itô's formula leads to

$$d[\Gamma\bar{Y}]_t = \langle \Gamma_{t^-}\bar{Z}_t + \bar{Y}_{t^-}\Gamma_{t^-}b_t, dW_t \rangle + \Gamma_{t^-} \int_{\mathcal{I}} \gamma_s^{Y_{s^-}^1, Z_s^1, U_s^2, U_s^1} \bar{U}_s(i)\tilde{\mu}(ds, di) - \Gamma_{t^-} d\hat{K}_t,$$

so that the process $\Gamma\bar{Y}$ is a supermartingale since $\Gamma > 0$. Hence

$$\Gamma_t \bar{Y}_t \geq \mathbb{E}\left[\Gamma_T \bar{Y}_T \middle| \mathcal{G}_t\right] = \mathbb{E}\left[\Gamma_T \bar{\xi} \middle| \mathcal{G}_t\right] \geq 0, \quad 0 \leq t \leq T,$$

leading to $\bar{Y} \geq 0$. □

We can now state our comparison results for the sequence $(Y^n)_n$.

Proposition 2.2.1 *Under* **(H0)**, *the sequence* $(Y^n)_n$ *is nondecreasing, and, for any quadruple* $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^{\mathbf{2}} \times \mathbf{L}_{\mathbb{G}}^{\mathbf{2}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^{\mathbf{2}}$ *satisfying* (2.2.1)-(2.2.2), *we have* $Y^n \leq \tilde{Y}$ *a.s.,* $n \in \mathbb{N}$. *Under additional Assumption* **(H1)**, *the sequence of processes* (Y^n) *converges increasingly and in* $\mathbf{L}_{\mathbb{G}}^{\mathbf{2}}(\mathbf{0}, \mathbf{T})$ *to a process* $Y \in \mathcal{S}_{\mathbb{G}}^{\mathbf{2}}$.

Proof. The monotonic property of the sequence (Y^n) follows from a direct application of Lemma 2.2.1 with $f_1 = f + n\int_{\mathcal{I}} h^- d\lambda$, $f_2 = f + (n+1)\int_{\mathcal{I}} h^- d\lambda$ and $K^2 \equiv 0$. For any quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^{\mathbf{2}} \times \mathbf{L}_{\mathbb{G}}^{\mathbf{2}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^{\mathbf{2}}$ satisfying (2.2.1)-(2.2.2), we obtain Y^n

$\leq \tilde{Y}$ a.s., $n \in \mathbb{N}$, taking $f_1 = f_2 = f + n\int_{\mathcal{I}} h^- d\lambda$ in the previous lemma. Under additional Assumption **(H1)**, the sequence (Y^n) is therefore increasing and upper bounded, ensuring its monotonic and in $\mathbf{L^2_{\mathbb{G}}(0, T)}$ convergence. □

We now turn to the convergence of the triplet $(Z^n, U^n, K^n)_n$ where, for any $n \in \mathbb{N}$, the increasing process $K^n \in \mathbf{A^2_{\mathbb{G}}}$ is defined by

$$K^n_t \;=\; n\int_0^t \int_{\mathcal{I}} h^-(s, Y^n_s, Z^n_s, U^n_s(i), i)\lambda(di)ds, \quad 0 \leq t \leq T.$$

For this purpose, we provide in Theorem 2.4.3 of the appendix an extension of Peng's monotonic limit theorem [64] to BSDEs with jumps.

Theorem 2.2.1 *Under* **(H0)**-**(H1)***, there exists a unique minimal solution* $(Y, Z, U, K) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L^2_{\mathbb{G}}(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2_{\mathbb{G}}}$ *to* (2.2.1)-(2.2.2)*, with K predictable. Furthermore Y is the increasing limit of $(Y^n)_n$, K is the weak limit of $(K^n)_n$ in* $\mathbf{L^2_{\mathbb{G}}(0, T)}$*, and*

$$\|Z^n - Z\|_{\mathbf{L^p(W)}} + \|U^n - U\|_{\mathbf{L^p}(\tilde{\mu})} \quad\longrightarrow\quad 0,$$

for any $p \in [1, 2)$, as n goes to infinity.

Proof. From Proposition 2.2.1 and Theorem 2.4.3, we derive the convergence of the sequence $(Y^n, Z^n, U^n, K^n)_n$ to (Y, Z, U, K), solution to (2.2.1). The constraint (2.2.2) is satisfied by (Y, Z, U, K) since the sequence $(K^n)_n$ is bounded in $\mathcal{S}^2_{\mathbb{G}}$, see (2.4.22) in the proof of Theorem 2.4.3. Observe also that K inherits the predictability of K^n, $n \in \mathbb{N}$. The uniqueness of the minimal solution (Y, Z, U, K) follows from the identification of the predictable, continuous and bounded variation parts. □

Remark 2.2.4 Observe that the purpose of Assumption **(H1)** is to ensure an upper bound on the sequence $(Y^n)_n$ of solutions to the penalized BSDEs. If such an upper bound already exists, this assumption is not required anymore but will be automatically satisfied from the existence of a minimal solution to the constrained BSDE with jumps. Cases where Assumption **(H1)** is satisfied are presented in the next section, and sufficient conditions for this assumption in a markovian setting are presented in [33].

Remark 2.2.5 Notice that the convergence of Y^n to Y, which is obtained in a weak sense (i.e. in $\mathbf{L^2_{\mathbb{G}}(0, T)}$) could be improved in the Markovian case (i.e. $Y^n \to Y$ in $\mathcal{S}^2_{\mathbb{G}}$). In this case we get the convergence (Z^n, U^n) to (Z, U) in $\mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$. See [33] for more details.

2.2.3 Link with multi-dimensional reflected Backward SDEs

In this section, we prove that multidimensional reflected BSDEs introduced by [44] and generalized by [41] are closely related to constrained BSDEs with jumps. The arguments presented here are purely probabilistic and therefore apply in the non Markovian framework considered in [44]. Furthermore, the proofs require precise comparison results based on viability properties that are reported in the Appendix for the convenience of the reader.

Recall that solving a multidimensional reflected BSDE consists in finding m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}_{\mathbb{F}}^{\mathbf{c},\mathbf{2}} \times \mathbf{L}_{\mathbb{F}}^{\mathbf{2}}(\mathbf{W}) \times \mathbf{A}_{\mathbb{F}}^{\mathbf{2}})^{\mathcal{I}}$ satisfying

$$\begin{cases} Y_t^i = \xi^i + \int_t^T \psi_i(s, Y_s^1, \ldots, Y_s^m, Z_s^i)ds - \int_t^T \langle Z_s^i, dW_s\rangle + K_T^i - K_t^i \\ Y_t^i \geq \max_{j\in A_i} h_{i,j}(t, Y_t^j) \\ \int_0^T [Y_t^i - \max_{j\in A_i}\{h_{i,j}(t, Y_t^j)\}]dK_t^i = 0 \end{cases} \tag{2.2.8}$$

where, for all $i \in \mathcal{I}$, $\psi_i :\ \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is an $\mathbb{F}$-progressively measurable map, $\xi^i \in \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, A_i is a nonempty subset of $\mathcal{I} \setminus \{i\}$, and , for any $j \in A_i$, $h_{i,j} : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given function. As detailed in [44], existence and uniqueness of a solution to (2.2.8) is ensured by the following assumption.

(H2)

(i) For any $i \in \mathcal{I}$ and $j \in A_i$, we have $\xi^i \geq h_{i,j}(T, \xi^j)$.

(ii) For any $i \in \mathcal{I}$, $\mathbb{E}\int_0^T \sup_{y\in\mathbb{R}^m|y_i=0} |\psi_i(t,y,0)|^2 dt + \mathbb{E}|\xi^i|^2 < +\infty$, and ψ_i is Lipschitz continuous: there exists a constant $k_\psi \geq 0$ such that

$$|\psi_i(t,y,z) - \psi_i(t,y',z')| \leq k_\psi(|y-y'| + |z-z'|)\ , \quad \forall (i,y,z,y',z') \in \mathcal{I} \times [\mathbb{R}\times\mathbb{R}^d]^2\ .$$

(iii) For any $i \in \mathcal{I}$, and $j \neq i$, ψ_i is increasing in its $(j+1)-$th variable i.e. for any $(t,y,y',z) \in \mathcal{I} \times [\mathbb{R}^m]^2 \times \mathbb{R}^d$ such that $y_k = y'_k$ for $k \neq j$ and $y_j \leq y'_j$ we have

$$\psi_i(t,y,z) \quad \leq \quad \psi_i(t,y',z) \quad \mathbb{P}-a.s.$$

(iv) For any $(i,t,y) \in \mathcal{I} \times [0,T] \times \mathbb{R}$ and $j \in A_i$, $h_{i,j}$ is continuous, $h_{i,j}(t,.)$ is a 1-Lipschitz and increasing function satisfying $h_{i,j}(t,y) \leq y$. Furthermore, for any $l \in A_j$, we have $l \in A_i \cup \{i\}$ and $h_{i,l}(t,y) > h_{i,j}(t, h_{j,l}(t,y))$.

Remark 2.2.6 Part (ii) and (iii) of Assumption **(H2)** are classical Lipschitz and monotony properties of the driver. Part (iv) ensures a tractable form for the domain of $\mathbb{R}^m$ where $(Y^i)_{i\in\mathcal{I}}$ lies, and (i) implies that the terminal condition is indeed in the domain,

Consider now the following constrained BSDE with jump: find a minimal quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}_{\mathbb{G}}^{\mathbf{2}} \times \mathbf{L}_{\mathbb{G}}^{\mathbf{2}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}_{\mathbb{G}}^{\mathbf{2}}$ satisfying

$$\begin{aligned} \tilde{Y}_t \quad &= \quad \xi^{I_T} + \int_t^T \psi_{I_s}(s, \tilde{Y}_s + \tilde{U}_s(1)\mathbf{1}_{I_s\neq 1}, \ldots, \tilde{Y}_s + \tilde{U}_s(m)\mathbf{1}_{I_s\neq m}, \tilde{Z}_s)ds \\ &\quad + \tilde{K}_T - \tilde{K}_t - \int_t^T \langle \tilde{Z}_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i)\mu(ds,di), \quad 0 \leq t \leq T,\ a.s. \end{aligned} \tag{2.2.9}$$

with

$$\mathbf{1}_{A_{I_{t^-}}}(i)\Big[\tilde{Y}_{t^-} - h_{I_{t^-},i}(t, \tilde{Y}_{t^-} + \tilde{U}_t(i))\Big] \;\geq\; 0, \qquad d\mathbb{P}\otimes dt \otimes \lambda(di) \;\; a.e. \tag{2.2.10}$$

where the process I is a pure jumps process defined by

$$I_t \;=\; I_0 + \int_0^t \int_{\mathcal{I}} (i - I_{s^-})\mu(ds, di)\,.$$

Remark that, if $\mu = \sum_{n\geq 0} \delta_{(\tau_n, A_n)}$, the process I is simply the pure jump process which coincides with A_n on each $[\tau_n, \tau_{n+1})$. This BSDE enters obviously into the class of constrained BSDEs with jumps of the form (2.2.1)-(2.2.2) studied above, with the following correspondence

$$\xi = \xi^{I_T}\;,\; f(t,y,z,u) = \psi_{I_t}(t, (y + u_i \mathbf{1}_{I_t \neq i})_{i\in\mathcal{I}}, z) \;\text{ and }\; h(t,y,z,v,i) = y - h_{I_{t^-},i}(t, y+v)\,.$$

Observe further that Assumption **(H0)** is automatically satisfied under **(H2)**, and, as detailed in the next proposition, **(H2)** also implies **(H1)** for (2.2.9)-(2.2.10) and its minimal solution can be directly related to the solution of the BSDE with oblique reflections (2.2.8).

Proposition 2.2.2 *Let Assumption* **(H2)** *hold and* $((Y^1, Z^1, K^1), \ldots, (Y^m, Z^m, K^m))$ *be the solution to* (2.2.8). *Then* **(H1)** *holds true for* (2.2.9)-(2.2.10) *and, if we denote* $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ *the minimal solution to* (2.2.9)-(2.2.10), *the following equality holds*

$$\tilde{Y}_t = Y_t^{I_t}, \quad \tilde{Z}_t = Z_t^{I_{t^-}} \quad \text{and} \qquad U_t(.) = Y_t^i - Y_{t^-}^{I_t}.$$

In order to derive this result, we need to introduce and discuss the corresponding penalized BSDEs. For $n \in \mathbb{N}$, consider the system of penalized BSDEs: find m couples $(Y^{i,n}, Z^{i,n})_{i\in\mathcal{I}} \in (\mathcal{S}_{\mathbb{F}}^{\mathbf{c},\mathbf{2}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^{\mathcal{I}}$ satisfying

$$\begin{aligned} Y_t^{i,n} \;&=\; \xi^i + \int_t^T \psi_i(s, Y_s^{1,n}, \ldots, Y_s^{m,n}, Z_s^{i,n})ds - \int_t^T \langle Z_s^{i,n}, dW_s\rangle \\ &\qquad + n\int_t^T \sum_{j\in A_i} [Y_s^{i,n} - h_{i,j}(s, Y_s^{j,n})]^- \lambda(j) ds, \quad 0 \leq t \leq T, \; a.s. \end{aligned} \tag{2.2.11}$$

Lemma 2.2.2 *Under* **(H2)**, *the sequence* $(Y^{i,n})_n$ *is increasing and converges to* Y^i $d\mathbb{P}\otimes dt$ *a.e. and in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, *and the sequence* $(Z^{i,n})_n$ *converges weakly to* Z^i *in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$, *for all* $i \in \mathcal{I}$.

Proof. For any $t \in [0, T]$, $y \in \mathbb{R}^m$ and $z \in [\mathbb{R}^d]^m$, set

$$\psi_i^n(t, y, z) \;\;:=\;\; \psi_i(t, y, z_i) + n\sum_{j\in A_i} [y_i - h_{i,j}(t, y_j)]^- \lambda(j).$$

From **(H2)** (iii) and (iv), we have $\psi_i^n(t, y + y', z) \geq \psi_i^n(t, y, z)$ for any $y' \in (\mathbb{R}^+)^m$ such that $y'_i = 0$. Since ψ_i^n depends only on z_i, it satisfies inequality (2.4.21) in the Appendix. Therefore, Theorem 2.4.2 leads to

$$Y_t^i \geq Y_t^{i,n+1} \geq Y_t^{i,n} \quad \text{for all} \quad (t, i, n) \in [0, T] \times \mathcal{I} \times \mathbb{N}\,. \tag{2.2.12}$$

By Peng's monotonic limit theorem, there exist m càdlàg processes $\hat{Y}^1, \ldots, \hat{Y}^m \in \mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$, $\hat{Z}^1, \ldots, \hat{Z}^m \in \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$ and $\hat{K}^1, \ldots, \hat{K}^m \in \mathbf{A}^{\mathbf{2}}_{\mathbb{F}}$, such that $Y^{i,n} \uparrow \hat{Y}^i$ a.e., $Y^{i,n} \to \hat{Y}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, $Z^{i,n} \to \hat{Z}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$ weakly and

$$\hat{Y}_t^i \;=\; \xi^i + \int_t^T \psi_i(s, \hat{Y}_s^1, \ldots, \hat{Y}_s^m, \hat{Z}_s^i) ds - \int_t^T \langle \hat{Z}_s^i, dW_s \rangle + \hat{K}_T^i - \hat{K}_t^i\,,$$

with $Y_t^i \geq \max_{j \in A_i} h_{i,j}(t, Y_t^j)$. Then, using the same arguments as in the proof of Theorem 2.4 in [41], we prove that $(\hat{Y}, \hat{Z}, \hat{K})$ is the unique solution to (2.2.8). □

Proof of Proposition 2.2.2. For $i \in \mathcal{I}$ and $n \in \mathbb{N}$, define the processes $Y^{I,n} \in \mathcal{S}^2_{\mathbb{G}}$, $Z^{I,n} \in \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W})$ and $U^{I,n} \in \mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ by

$$Y_t^{I,n} \;:=\; Y_t^{I_t,n}\,, \quad Z_t^{I,n} \;:=\; Z_t^{I_{t-},n} \quad \text{and} \quad U_s^{I,n}(i) \;:=\; Y_s^{i,n} - Y_{s^-}^{I,n}\,, \quad 0 \leq t \leq T. \tag{2.2.13}$$

We deduce from (2.2.11) that $(Y^{I,n}, Z^{I,n}, U^{I,n})$ is the solution to the penalized BSDE associated to (2.2.9)-(2.2.10) given by

$$\begin{aligned} Y_t^{I,n} \;&=\; \xi^{I_T} + \int_t^T \psi_{I_s}(s, Y_{s^-}^{I,n} + U_s^{I,n}(1)\mathbf{1}_{I_s \neq 1}, \ldots, Y_{s^-}^{I,n} + U_s^{I,n}(m)\mathbf{1}_{I_s \neq m}, Z_s^{I,n}) ds \\ &\quad - \int_t^T \langle Z_s^{I,n}, dW_s \rangle + n \int_t^T \!\! \int_{\mathcal{I}} h^-(s, Y_{s^-}^{I,n}, Z_s^{I,n}, U_s^{I,n}(i)) \lambda(i) ds + \int_t^T \!\! \int_{\mathcal{I}} U_s^{I,n}(i) \mu(di, ds)\,. \end{aligned}$$

From (2.2.12), the sequence $(Y^{I,n})_n$ is bounded in $\mathcal{S}^2_{\mathbb{G}}$ and, proceeding as in Section 2, we prove that

$$\|Y^{I,n} - \tilde{Y}\|_{\mathbf{L^2(0,T)}} + \|Z^{I,n} - \tilde{Z}\|_{\mathbf{L^p(W)}} + \|U^{I,n} - \tilde{U}\|_{\mathbf{L^p}(\tilde{\mu})} \;\longrightarrow\; 0\,,$$

where $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is the minimal solution to (2.2.9)-(2.2.10). Combined with (2.2.13) and Lemma 2.2.2, this concludes the proof. □

The main interest of the previous result is the unification of the notion of constrained BSDEs without jumps studied in [65], the class of BSDEs with constrained jumps introduced by [46] and the notion of BSDEs with oblique reflections considered in [41] and [44]. As a by product, we deduce from [44] that constrained BSDE with jump of the form interpret as a viscosity solution to a system of variational inequalities. This opens the door to the numerical approximation of solution to this type of PDEs by purely probabilistic schemes in the spirit of the algorithm presented in [12]. Instead of considering a multidimensional BSDE with oblique reflections, one just needs to approximate a one dimensional constrained BSDE with jumps. The Feynman-Kac representation of general constrained BSDE with jumps and the corresponding numerical algorithm are under study and will appear in a separate paper [33].

Since the results presented here rely on probabilistic arguments, they can apply to eventually non Markovian settings considered in [41] or [44]. Nevertheless, they do not include cases where the dynamics of the underlying diffusion depends on the value of the current switching regime. The purpose of the next section is to extend the link presented here to a more general class of non-Markovian switching problems.

2.3 Constrained Backward SDEs with jumps and non-Markovian switching

This section is devoted to the interpretation of non-Markovian switching problems in terms of solutions to BSDEs with constrained jumps. In particular, we consider useful cases where the current switching regime influences the dynamics of the underlying diffusion. This is the case for example if we consider a producer who is a large investor on the commodity market who influences the dynamics of the underlying commodity prices. One of the dimension of the underlying can also be the level of the stock in some commodity which is of course directly related to the chosen mode of production. To our knowledge, no BSDE representation has yet been established in this type of framework. We first extend the results of [44] and relate the solution to a general non-Markovian switching problem with a well chosen family of multidimensional BSDE with oblique reflections. We finally link this family of BSDE with one single one-dimensional constrained BSDE with jumps leading to the announced representation property.

2.3.1 Non-Markovian optimal switching

Given the set $\mathcal{I} = \{1, \ldots, m\}$ and a terminal time $T < +\infty$, an impulse strategy α consists in a sequence $\alpha := (\tau_k, \zeta_k)_{k\geq 1}$, where $(\tau_k)_{k\geq 1}$ is an increasing sequence of $\mathbb{F}$-stoppping times, and ζ_i are $\mathcal{F}_{\tau_i}$-measurable random variables valued in $\mathcal{I}$. To a strategy $\alpha = (\tau_k, \zeta_k)_{k\geq 1}$ and an initial regime i_0, we naturally associate the process $(\alpha_t)_{t\leq T}$ defined by

$$\alpha_t \quad := \quad \sum_{k\geq 0} \zeta_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t) \,.$$

with $\tau_0 = 0$ and $\zeta_0 = i_0$. We denote by $\mathcal{A}$ the set of admissible strategies. Given a strategy $\alpha \in \mathcal{A}$ and an initial condition (i_0, X_0), we define the controlled process X^α by

$$X_t^\alpha = X_0 + \int_0^t b(s, \alpha_s, X_s^\alpha) ds + \int_0^t \sigma(s, \alpha_s, X_s^\alpha) dW_s \,, \tag{2.3.1}$$

and we consider the total profit at horizon T defined by

$$J(\alpha) \quad := \quad \mathbb{E}\left[g(\alpha_T, X_T^\alpha) + \int_0^T \psi(s, \alpha_s, X_s^\alpha)ds + \sum_{0<\tau_k\leq T} c(\tau_k, \zeta_{k-1}, \zeta_k)\right] . \qquad (2.3.2)$$

We suppose here that the functions $b, \sigma, \psi : \ \Omega \times [0,T] \times \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d\times d}, \mathbb{R}$ are progressively measurable functions and $g : \ \Omega \times \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{F}_T \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and that $c : \ \Omega \times [0,T] \times \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ is a progresively measurable function.

Given an initial data (X_0, i_0), the switching problem consists in finding a strategy $\alpha^* \in \mathcal{A}$ such that

$$J(\alpha^*) = \sup_{\alpha\in\mathcal{A}} J(\alpha)$$

Such a strategy is called optimal and we shall work under the following assumption.

(H3)

(i) b and σ satisfy the Lipschitz property: there exists a constant k such that $\mathbb{P}-$a.s.

$$|b(\omega,t,i,x) - b(\omega,t,i,x')| + |\sigma(\omega,t,i,x) - \sigma(\omega,t,i,x')| \quad \leq \quad k\,|x - x'|\,,$$

for all $(\omega, t, i, x, x') \in \Omega \times [0,T] \times \mathcal{I} \times \mathbb{R}^d \times \mathbb{R}^d$.

(ii) The terminal condition g satisfies the following structural condition

$$g(\omega,x,i) \geq \max_{j\in\mathcal{I}}\{g(\omega,x,j) + C(\omega,T,i,j)\} \quad \forall(\omega,i,x) \in \Omega \times \mathbb{R}^d \times \mathcal{I},$$

(iii) The functions g and ψ are bounded, i.e. there exists two constants $\bar{g}$ and $\bar{\psi}$ satisfying

$$\sup_{(\omega,i,x)\in\Omega\times\mathcal{I}\times\mathbb{R}^d}\{|g(\omega,i,x)|\} \leq \bar{g} \quad \text{and} \quad \sup_{(\omega,t,i,x)\in\Omega\times[0,T]\times\mathcal{I}\times\mathbb{R}^d}\{|\psi(\omega,t,i,x)|\} \leq \bar{\psi}\,.$$

(iv) The cost function c is upper-bounded, i.e. there exists a constant $\bar{c} > 0$ such that

$$\max_{(\omega,t,i,j)\in\Omega\times[0,T]\times\mathcal{I}\times\mathcal{I}} c(\omega,t,i,j) \leq -\bar{c}\,.$$

Furthermore $c(.,i,j)$ is continuous, for all $i,j \in \mathcal{I}$, and it satisfies the structural condition

$$c(\omega,t,i,l) > c(\omega,t,i,j)+c(\omega,t,j,l)\,, \quad \forall(\omega,t,i,j,l) \in \Omega\times[0,T]\times[\mathcal{I}]^3 \text{ s.t. } j \neq i\,,\ j \neq l\,.$$

Remark 2.3.1 Part (i) of Assumption **(H3)** provides existence and uniqueness of a solution to (2.3.1). Part (ii) ensures the non-optimality of a switching at maturity, (iv) makes indirect switching strategy irrelevant and (iii)-(iv) ensures the problem is well posed.

Let define the set of finite strategies $\mathcal{D}$ by

$$\mathcal{D} \quad := \quad \{\alpha = (\tau_k, \zeta_k)_{k\geq 1} \in \mathcal{A} \mid \mathbb{P}(\tau_k < T,\ \ \forall k \geq 1) = 0\}\ .$$

Let first observe the following property:

Proposition 2.3.1 *Under* ***(H3)****, the supremum of J over $\mathcal{A}_{0,i}$ coincides with the one over $\mathcal{D}_{0,i}$, that is*

$$\sup_{\alpha\in\mathcal{A}} J(\alpha) = \sup_{\alpha\in\mathcal{D}} J(\alpha)\,, \qquad \forall i\in\mathcal{I}\,. \tag{2.3.3}$$

Proof. Fix $i \in \mathcal{I}$. Consider a strategy $\alpha = (\tau_k, \zeta_k)_{k\geq 0} \in \mathcal{A}\setminus\mathcal{D}$ and define $B := \{\omega \in \Omega \mid \tau_n(\omega) < T,\ \forall n \in \mathbb{N}^*\}$ so that $\mathbb{P}(B) > 0$. Such a strategy is not optimal since we derive from **(H3)** (iii) and (iv) that

$$J(\alpha) \quad\leq\quad \bar g + T\bar\psi + \mathbb{E}\left[\sum_{0<\tau_k\leq T} c(\tau_k,\zeta_{k-1},\zeta_k)\mathbf{1}_B\right] + \mathbb{E}\left[\sum_{0<\tau_k\leq T} c(\tau_k,\zeta_{k-1},\zeta_k)\mathbf{1}_{B^c}\right] \quad=\quad -\infty\,.$$

□

2.3.2 Reflected BSDEs and optimal switching

Following the approach of [27], we consider in this section a family of reflected BSDE. For any couple (ν,η) with ν a stopping time valued in $[0,T]$ and η a $\mathcal{F}_\nu$-measurable random variable taking values in $\mathbb{R}^d$, we consider the following reflected BSDE

$$\left\{\begin{array}{l}
(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}} \in (\mathcal{S}^2_{\mathbb{F}}\times\mathbf{L}^2_{\mathbb{F}}(\mathbf{W})\times\mathbf{A}^2_{\mathbb{F}})^{\mathcal{I}},\\
Y^{\nu,i,\eta}_t = g(i,X^{\nu,i,\eta}_T) + \int_t^T \psi(s,i,X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu}ds - \int_t^T\langle Z^{\nu,i,\eta}_s, dW_s\rangle + K^{\nu,i,\eta}_T - K^{\nu,i,\eta}_t,\\
Y^{\nu,i,\eta}_t \geq \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t + c(t,i,j)\},\\
\int_0^T [Y^{\nu,i,\eta}_t - \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t + c(t,i,j)\}]dK^{\nu,i,\eta}_t \;=\; 0,
\end{array}\right. \tag{2.3.4}$$

where $X^{\nu,i,\eta}$ is the diffusion defined by

$$X^{\nu,i,\eta}_t \quad=\quad \eta\mathbf{1}_{t\geq\nu} + \int_0^t b(s,i,X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu}ds + \int_0^t \sigma(s,i,X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu}dW_s\,, \qquad \forall t\geq 0 \tag{2.3.5}$$

Under **(H3)**, we know from [41] that (2.3.4) has a unique solution for any stopping time ν and any $\mathcal{F}_\nu$-measurable random variable η, and we denote by $\mathcal{O}^{\nu,\cdot,\eta}$ its barrier defined by

$$O^{\nu,i,\eta}_t \;:=\; \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t + c(t,i,j)\}\,, \qquad i\in\mathcal{I}\,,\;\; t\leq T. \tag{2.3.6}$$

We aim at relating the solutions to this class of reflected BSDEs to the solution of the optimal non Markovian switching problem presented in (2.3.2). The next proposition relates a stability property, a Snell envelope representation and a global estimate on the family of processes $(Y^{\nu,\cdot,\eta})_{(\nu,\eta)}$.

Proposition 2.3.2 *Assume that* ***(H3)*** *holds and take ν,ν' two $\mathbb{F}$-stopping times such that $\nu\leq\nu'$ and η an $\mathcal{F}_\nu$-measurable random variable valued in $\mathbb{R}^d$.*

(i) For all $i\in\mathcal{I}$ and $t\geq\nu'$, we have $Y^{\nu,i,\eta}_t = Y^{\nu',i,X^{\nu,i,\eta}_{\nu'}}_t$, $\mathbb{P}$-a.s.

(ii) For all $i \in \mathcal{I}$ and $t \geq \nu'$, we have the following representation

$$\begin{aligned} Y_t^{\nu,i,\eta} &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\Big[\int_t^\tau \psi(s,i,X_s^{\nu,i,\eta})\mathbf{1}_{s\geq\nu}ds \\ &\qquad +\mathcal{O}_\tau^{\nu,i,\eta}\mathbf{1}_{\tau<T} + g(i,X_T^{\nu,i,\eta})\mathbf{1}_{\tau=T}\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{2.3.7}$$

(iii) There exists a constant $\bar{Y}$ such that

$$\sup_{(t,\nu,i,\eta)} |Y_t^{\nu,i,\eta}| \leq \bar{Y}. \tag{2.3.8}$$

Proof. (i) Notice first that $X^{\nu,i,\eta}$ and $X^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}}$ solve the same SDE, namely

$$X_{\nu'} = X_{\nu'}^{\nu,i,\eta} \quad \text{and} \quad dX_t = b(t,i,X_t)dt + \sigma(t,i,X_t)dW_t \quad \text{for } t \geq \nu'. \tag{2.3.9}$$

Under **(H3)** (i), equation (2.3.9) admits a unique solution and we have $X^{\nu',i,X_{\nu'}^{\nu,i,\eta}} = X^{\nu,i,\eta}$ on $[\nu',T]$. We deduce that $(Y^{\nu',i,X_{\nu'}^{\nu,i,\eta}}, Z^{\nu',i,X_{\nu'}^{\nu,i,\eta}}, K^{\nu',i,X_{\nu'}^{\nu,i,\eta}})_{i\in\mathcal{I}}$ satisfies the same BSDE as $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}}$ on $[\nu',T]$. Uniqueness of solution to this BSDE is given by [41].

(ii) Regarding of (2.3.4), $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})$ interprets as the solution to a reflected BSDE with single barrier $\mathcal{O}^{\nu,i,\eta}$. We deduce from [29] that $Y^{\nu,i,\eta}$ admits the snell envelope representation (2.3.7).

(iii) For fixed ν and η, the family $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}}$ is the solution to the BSDE with oblique reflection (2.3.4). We know from [41] that $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})$ converges to $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})$, where the sequence $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})_n$ is defined recursively by

$$Y_t^{\nu,i,\eta,0} = g(i,X_T^{\nu,i,\eta}) + \int_t^T \psi(s,i,X_T^{\nu,i,\eta})\mathbf{1}_{s\geq\nu}ds - \int_t^T \langle Z_s^{\nu,i,\eta,0}, dW_s\rangle \quad \text{and} \quad K_t^{\nu,i,\eta,0} = 0,$$

and, for $n \geq 1$,

$$\begin{cases} Y_t^{\nu,i,\eta,n} = g(i,X_T^{\nu,i,\eta}) + \int_t^T \psi(s,i,X_s^{\nu,i,\zeta})\mathbf{1}_{s\geq\nu}ds \\ \qquad\qquad - \int_t^T \langle Z_t^{\nu,i,\eta,n}, dW_s\rangle + K_T^{\nu,i,\eta,n} - K_t^{\nu,i,\eta,n}, \\ Y_t^{\nu,i,\eta,n} \geq \max_{j\in\mathcal{I}}\{Y_t^{\nu,j,\eta,n-1} + c(t,i,j)\}, \\ \int_0^T [Y_t^{\nu,i,\eta,n} - \max_{j\in\mathcal{I}}\{Y_t^{\nu,j,\eta,n-1} + c(t,i,j)\}]dK_t^{\nu,i,\eta,n} = 0. \end{cases} \tag{2.3.10}$$

To derive (2.3.8), it suffices to prove by induction on n that

$$|Y_t^{\nu,i,\eta,n}| \leq (T-t+1)\max\{\bar{\psi},\bar{g}\}, \quad i \in \mathcal{I}, \ 0 \leq t \leq T, \ n \in \mathbb{N}.$$

First, rewriting $Y^{\nu,i,\eta,0}$ as a conditional expectation, we derive

$$|Y_t^{\nu,i,\eta,0}| \leq (T-t)\bar{\psi} + \bar{g} \leq (T-t+1)\max\{\bar{\psi},\bar{g}\}, \qquad 0 \leq t \leq T, \ i \in \mathcal{I}.$$

Fix $n \in \mathbb{N}$ and suppose the result is true for $Y^{\cdot,n}$. Using the representation of $Y^{\nu,i,\eta,n+1}$ as a Snell envelope, we derive

$$Y_t^{\nu,i,\eta,n+1} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi(s,i,X_s^{\nu,i,\eta})\mathbf{1}_{s\geq\nu}ds + \mathcal{O}_\tau^{\nu,i,\eta,n+1}\mathbf{1}_{\tau<T} + g(i,X_T^{\nu,i,\eta})\mathbf{1}_{\tau=T}\middle|\mathcal{F}_t\right],$$

where $\mathcal{O}_\tau^{\nu,i,\eta,n+1} := \max_{j\in\mathcal{I}}\{Y_\tau^{\nu,j,\eta,n} + c(t,i,j)\}$. Combining this representation with Assumption **(H3)** leads to $|Y_t^{\nu,i,\eta,n+1}| \leq (T-t+1)\max\{\bar{g},\bar{\psi}\}$ and concludes the proof. □

For any stopping time ν, any $\mathcal{F}_\nu$-random variable η and any $\mathcal{I}$-valued random variable ζ, we naturally introduce the processes $Y^{\nu,\zeta,\eta}$ and $\mathcal{O}^{\nu,\zeta,\eta}$ by

$$Y_t^{\nu,\zeta,\eta} = \sum_{i\in\mathcal{I}} Y_t^{\nu,i,\eta}\mathbf{1}_{\zeta=i} \quad \text{and} \quad \mathcal{O}_t^{\nu,\zeta,\eta} = \sum_{i\in\mathcal{I}} \mathcal{O}_t^{\nu,i,\eta}\mathbf{1}_{\zeta=i}. \tag{2.3.11}$$

We are now able to state the main results of this section characterizing the optimal solution to the switching problem (2.3.2) in terms of reflected BSDEs.

Theorem 2.3.1 *Let $\alpha^* = (\tau_n^*,\zeta_n^*)_{n\geq 0}$ be the strategy given by $\alpha_0^* = (0,i_0)$ and defined recursively for $n \geq 1$ by*

$$\tau_n^* := \inf\left\{s \geq \tau_{n-1}^* \ ; \ Y_s^{\tau_{n-1}^*,\zeta_{n-1}^*,X^*_{\tau_{n-1}^*}} = \mathcal{O}_s^{\tau_{n-1}^*,\zeta_{n-1}^*,X^*_{\tau_{n-1}^*}}\right\}, \tag{2.3.12}$$

$$\zeta_n^* \ \textit{is s.t.} \ \mathcal{O}_{\tau_n^*}^{\tau_{n-1}^*,\zeta_{n-1}^*,X^*_{\tau_{n-1}^*}} = Y_{\tau_n^*}^{\tau_n^*,\zeta_n^*,X^*_{\tau_n^*}} + c(\tau_n^*,\zeta_{n-1}^*,\zeta_n^*), \tag{2.3.13}$$

with X^* the diffusion defined by

$$X_t^* = x_0 + \sum_{n\geq 1}\int_{\tau_{n-1}^*}^{\tau_n^*} b(s,\zeta_{n-1}^*,X_s^*)\mathbf{1}_{s\leq t}ds + \int_{\tau_{n-1}^*}^{\tau_n^*}\sigma(s,\zeta_{n-1}^*,X_s^*)\mathbf{1}_{s\leq t}ds, \quad t\geq 0.$$

*Under Assumption **(H3)**, the strategy α^* is optimal for the switching problem* (2.3.2) *and we have*

$$Y_0^{i_0}(0,x_0) = J(\alpha^*). \tag{2.3.14}$$

Proof. The proof is performed in two steps.

Step 1. *The strategy $\alpha^* \in \mathcal{D}$ and satisfies $Y_0^{0,i_0,x_0} = J(\alpha^*)$.*
The representation (2.3.7) rewrites

$$Y_0^{0,i_0,x_0} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0}\mathbb{E}\left[\int_0^\tau \psi(s,i_0,X_s^{0,i_0,x_0})ds + \mathcal{O}_\tau^{0,i_0,x_0}\mathbf{1}_{\tau<T} + g(i_0,X_T^{0,i_0,x_0})\mathbf{1}_{\tau=T}\right].$$

Since the boundary $\mathcal{O}^{0,i_0,x_0}$ is continuous, the stopping time τ_1^* is optimal for (2.3.15) and we get

$$Y_0^{0,i_0,x_0} = \mathbb{E}\left[\int_0^{\tau_1^*}\psi(s,i_0,X_s^{0,i_0,x_0})ds + \mathcal{O}_{\tau_1^*}^{0,i_0,x_0}\mathbf{1}_{\tau_1^*<T} + g(i_0,X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1^*=T}\right].$$

If $\tau_1^* = T$, the proof is over. Let suppose that $\tau_1^* < T$ and, according to the definition of ζ_1^*, we derive

$$Y_0^{0,i_0,x_0} \quad = \quad \mathbb{E}\left[\int_0^{\tau_1^*} \psi(s, i_0, X_s^{0,i_0,x_0})ds + Y_{\tau_1^*}^{\tau_1^*,\zeta_1^*,X_{\tau_1^*}^{0,i_0,x_0}} + c(\tau_1^*, i_0, \zeta_1^*)\right] .$$

Similarly, we can use the representation of $Y_{\tau_1^*}^{\tau_1^*,\zeta_1^*,X_{\tau_1^*}^{0,i_0,x_0}}$ given by (2.3.7), and we deduce recursively that

$$Y_0^{0,i_0,x_0} \quad = \quad \mathbb{E}\left[\int_0^{\tau_n^*} \psi(s, i_0, X_s^*)ds + Y_{\tau_n^*}^{\tau_n^*,\zeta_n^*,X_{\tau_n^*}^*} + \sum_{0<k\le n} c(\tau_k, \zeta_{k-1}^*, \zeta_k^*)\right] , \tag{2.3.15}$$

for all $n \in \mathbb{N}$ satisfying $\tau_n < T$. We now prove $\alpha^* \in \mathcal{D}$ and assume on the contrary that $p := \mathbb{P}(\tau_n^* < T, \ \forall n \in \mathbb{N}) > 0$. Combining **(H3)**, (2.3.8) and (2.3.15), we derive

$$Y_0(0, i_0, x_0) \ \le \ \bar{\psi}T + \mathbb{E}\left[\sup_{s\le T}\left|Y_s^{\tau_n^*,\zeta_n^*,X_{\tau_n^*}^*}\right|\right] - n\bar{c}\mathbb{P}(\tau_k^* < T \ , \ \forall k \ge 0) \ \le \ \bar{\psi}T + \bar{Y} - n\bar{c}p\,.$$

Sending n to $-\infty$ leads to $Y_0^{0,i_0,x_0} = -\infty$ which contradicts $Y^{0,i_0,x_0} \in \mathcal{S}^2_{\mathbb{F}}$. Therefore $\mathbb{P}(\tau_k^* < T \ , \ \forall k \ge 0) = 0$ i.e. $\alpha^* \in \mathcal{D}$. Finally, taking the limit as $n \to \infty$ in (2.3.15) leads to $Y_0^{0,i_0,x_0} = J(\alpha^*)$.

Step2. *The strategy α^* is optimal.*
According to Proposition 2.3.1, it suffices to consider finite strategies and we pick any $\alpha = (\tau_n, \zeta_n)_{n\ge 0} \in \mathcal{D}$. Since τ_1^* is optimal, we deduce from Part (i) of Proposition 2.3.2 that

$$\begin{aligned}
Y_0^{0,i_0,x_0} \quad &- \quad \mathbb{E}\left[\int_0^{\tau_1} \psi(s, i_0, X_s^{0,i_0,x_0})ds\right] \\
&\ge \quad \mathbb{E}\left[\mathcal{O}_{\tau_1}^{0,i_0,x_0}\mathbf{1}_{\tau_1<T} + g(i_0, X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1=T}\right] \\
&\ge \quad \mathbb{E}\left[(Y_{\tau_1}^{0,\zeta_1,x_0} + c(\tau_1, i_0, \zeta_1))\mathbf{1}_{\tau_1<T} + g(i_0, X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1=T}\right] \\
&\ge \quad \mathbb{E}\left[\left(Y_{\tau_1}^{\tau_1,\zeta_1,X_{\tau_1}^{0,\zeta_1,x_0}} + c(\tau_1, i_0, \zeta_1)\right)\mathbf{1}_{\tau_1<T} + g(i_0, X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1<T}\right] .
\end{aligned}$$

Proceeding as in step 1, an induction argument leads to

$$Y_0^{0,i_0,x_0} \ge \mathbb{E}\Big[\int_0^{\tau_n} \psi(i_0, X_s^\alpha)ds + Y_{\tau_n}^{\tau_n,\zeta_n,X_{\tau_n}^\alpha}\mathbf{1}_{\tau_n<T} + g(\zeta_n, X_T^\alpha)\mathbf{1}_{\tau_n=T} + \sum_{0<k\le n} c(\tau_k, \zeta_{k-1}, \zeta_k)\Big] ,$$

for all n satisfying $\tau_n < T$. Since the strategy α is finite, we deduce $Y_0^{0,i_0,x_0} \ge J(\alpha)$ by sending $n \to \infty$. $\square$

2.3.3 Approximation by penalisation and link with constrained BSDEs with jumps

We finally conclude the paper and present in this paragraph the link between constrained Backward SDEs with jumps and optimal switching in non-Markovian cases.

Consider the constrained BSDE with jumps: find a quadruple $(Y, Z, U, K) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying

$$\begin{aligned} Y_t &= g(I_T, X_T^I) + \int_t^T \psi(s, I_s, X_s^I)ds + K_T - K_t \\ &\quad - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds, di), \quad 0 \le t \le T, \ a.s. \end{aligned} \tag{2.3.16}$$

with

$$-U_t(i) - c(t, I_{t^-}, i) \ \ge \ 0, \qquad d\mathbb{P} \otimes dt \otimes \lambda(di) \ \ a.e. \tag{2.3.17}$$

where the process (I, X^I) is defined by

$$\begin{aligned} I_t &= i_0 + \int_0^t \int_{\mathcal{I}} (i - I_{t^-})\mu(dt, di) \ , \\ X_t^I &= x_0 + \int_0^t b(s, I_s, X_s^I)ds + \int_0^t \sigma(s, I_s, X_s^I)dWs \ , \end{aligned}$$

for $t \ge 0$. The link between (2.3.16)-(2.3.17) and the optimal switching problem is given by the following result which extends the link between BSDEs with constrained jumps and BSDEs with oblique reflections presented in Proposition 2.2.2.

Proposition 2.3.3 *Under* ***(H3)***, ***(H1)*** *holds for* (2.3.16)-(2.3.17) *and if we denote* (Y, Z, U, K) *its minimal solution we have*

$$Y_t = Y_t^{t, I_t, X_t^I}, \quad Z_t = Z_t^{t, I_{t^-}, X_t^I} \quad \text{ and } \quad U_t(i) = Y_t^{t, I_t, X_t^I} - Y_{t^-}^{t, I_{t^-}, X_t^I}. \tag{2.3.18}$$

for $0 \le t \le T$ *a.s.. In particular, we deduce* $Y_0 = J(\alpha^*) = \sup_{\alpha \in \mathcal{A}} J(\alpha)$.

Proof. For any stopping time ν and any random variable η, let define the processes $(\tilde{Y}^{\nu,i,\eta,n}, \tilde{Z}^{\nu,i,\eta,n}, \tilde{K}^{\nu,i,\eta,n})_{i \in \mathcal{I}} \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^{\mathcal{I}}$ as the solution to the penalized BSDE

$$\begin{aligned} \tilde{Y}_t^{\nu,i,\eta,n} &= g(i, X_T^{\nu,i,\eta}) + \int_t^T \psi(s, i, X_s^{\nu,i,\eta})ds - \int_t^T \langle \tilde{Z}_s^{\nu,i,\eta,n}, dWs\rangle \\ &\quad + n\int_t^T \Big\{ \sum_{j \in \mathcal{I}} [\tilde{Y}_s^{\nu,j,\eta,n} + c(s,i,j) - \tilde{Y}_s^{\nu,i,\eta,n}]^- \lambda(j) \Big\} ds \end{aligned}$$

Under **(H3)**, we know from Lemma 2.2.2 (or [44]) that $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})_{i \in \mathcal{I}}$ converges to $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i \in \mathcal{I}}$ as n goes to ∞, for each (ν, η). Proceeding as in the proof of Proposition 2.2.2, one easily checks that the quadruple $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$ defined by

$$\tilde{Y}_t^n = \tilde{Y}_t^{t, I_t, X_t^I, n}, \quad \tilde{Z}_t^n = \tilde{Z}_t^{t, I_{t^-}, X_t^I, n} \quad \text{ and } \quad \tilde{U}_t^n(i) = \tilde{Y}_t^{t, I_t, X_t^I, n} - \tilde{Y}_{t^-}^{t, I_{t^-}, X_t^I, n}$$

is solution to the penalized BSDE associated to (2.3.16)-(2.3.17), namely

$$\begin{aligned} Y_t &= g(I_T, X_T^I) + \int_t^T \psi(s, I_s, X_s^I)ds + n\int_t^T \int_{\mathcal{I}} [U_s(i) + c(s, I_{s^-}, i)]^- \lambda(di)ds \\ &\quad - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds, di), \quad 0 \le t \le T. \ a.s. \end{aligned}$$

From Proposition 2.3.2 (iii), we know that the monotone sequence $(\tilde{Y}^n)_n$ is bounded, and we derive from Remark 2.2.4 that Assumption **(H1)** is satisfied and that $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$ converges to (Y, Z, U), which concludes the proof. □

Remark 2.3.2 The optimal strategy can be described by the constrained BSDE with jumps (2.3.16)-(2.3.17). Indeed, using the definition of $(\tau_n^*, \zeta_n^*)_{n\ge 0}$ and the identification (2.3.18) we get

$$\tau_{n+1}^* = \inf\left\{ t \ge \tau_n^* \ ; \ \max_{j\in\mathcal{I}} \mathbb{E}\left[U_t(j) - c(t, \zeta_n^*, j) \Big| I_s = \zeta_n^* \ \ \forall s \ge \tau_n^* \right] = 0 \right\}$$

and ζ_{n+1} such that

$$\mathbb{E}\left[U_{\tau_{n+1}^*}(\zeta_{n+1}^*) - c(\tau_{n+1}^*, \zeta_n^*, \zeta_{n+1}^*)] \Big| I_s = \zeta_n^* \ \ \forall s \ge \tau_n \right] = 0.$$

2.4 Appendix

2.4.1 Viability property for BSDEs

We extend here the viability property of [17] for a closed convex cone $\mathcal{C}$ of $\mathbb{R}^m$. Let (Y, Z) $\in (\mathcal{S}_{\mathbb{F}}^{c,2} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^m$ satisfying

$$Y_t = Y_T + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T \langle Z_s, dW_s\rangle + K_T - K_t, \qquad 0 \le t \le T,$$

where $F : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is a progressively measurable function satisfying **(H2)** (i) and (ii) and K is an $\mathbb{R}^m$-valued finite variation process such that

$$K_t = \int_0^t k_s d|K|_s,$$

with $k_t \in \mathcal{C}$ and $|K|_s$ the variation of K on $[0, s]$. Denote $d_{\mathcal{C}}$ the distance to $\mathcal{C}$ (i.e. $d_{\mathcal{C}}(x) = \min_{y\in\mathcal{C}} |x - y|$) and $\Pi_{\mathcal{C}}$ the projection operator onto $\mathcal{C}$, then we have the following result:

Proposition 2.4.4 *Suppose $Y_T \in \mathcal{C}$ and there exists a constant C^0 such that F satisfies*

$$4\langle y - \Pi_{\mathcal{C}}(y), F(t, y, z)\rangle \le \langle D^2|d_{\mathcal{C}}|^2(y)z, z\rangle + 2C^0|d_{\mathcal{C}}|^2(y) \tag{2.4.19}$$

for any point $y \in \mathbb{R}^m$ where $|d_{\mathcal{C}}|^2$ is twice differentiable. Then, we have

$$Y_t \in \mathcal{C}, \qquad \text{for all } t \in [0, T] \quad \mathbb{P} - a.s.$$

Proof. Let $\eta \in C^\infty(\mathbb{R}^d)$ be a non-negative function with support in the unit ball and such that $\int_{\mathbb{R}^d} \eta(x)dx = 1$. For $\delta > 0$ and $x \in \mathbb{R}^d$ we put

$$\begin{aligned} \eta_\delta(x) &:= \frac{1}{\delta^d}\eta\Big(\frac{x}{\delta}\Big) \\ \phi_\delta(x) &:= |d_{\mathcal{C}}|^2 \star \eta_\delta(x) \; = \; \int_{\mathbb{R}^d} |d_{\mathcal{C}}(x-x')|^2 \eta_\delta(x')dx' \end{aligned}$$

Notice (see part (b) of the proof of Theorem 2.5 in [17]) that $\phi_\delta \in C^\infty(\mathbb{R}^d)$ and

$$\begin{cases} 0 \;\le\; \phi_\delta(x) \;\le\; (d_{\mathcal{C}}(x)+\delta)^2 \\ D\phi_\delta(x) \;=\; \int_{\mathbb{R}^d} D|d_{\mathcal{C}}(x')|^2\eta_\delta(x-x')dx' \;\text{ and } |D\phi_\delta(x)| \;\le\; 2(d_{\mathcal{C}}(x)+\delta) \\ D^2\phi_\delta(x) \;=\; \int_{\mathbb{R}^d} D^2|d_{\mathcal{C}}(x')|^2\eta_\delta(x-x')dx' \text{ and } 0 \;\le\; |D^2\phi_\delta(x)| \;\le\; 2I_d \end{cases} \tag{2.4.20}$$

Applying Itô's formula to $\phi_\delta(Y_t)$, this leads to

$$\begin{aligned} \mathbb{E}\phi_\delta(Y_t) \;&=\; \mathbb{E}\phi_\delta(Y_T) + \mathbb{E}\int_t^T \langle D\Phi_\delta(Y_s), F(s,Y_s,Z_s)\rangle ds - \frac{1}{2}\mathbb{E}\int_t^T \langle \Phi_\delta(Y_s)Z_s, Z_s\rangle ds \\ &\quad + \mathbb{E}\int_t^T \langle D\Phi_\delta(Y_s), k_s\rangle d|K|_s \\ &\le\; \delta^2 + \mathbb{E}\int_t^T\!\!\int_{\mathbb{R}^d} \Big[\langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s)\rangle - \frac{1}{2}\langle D^2|d_{\mathcal{C}}(y)|^2 Z_s, Z_s\rangle\Big]\eta_\delta(Y_s-y)dyds \\ &\quad - \mathbb{E}\int_t^T\!\!\int_{\mathbb{R}^d} \langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s)-F(s,Y_s,Z_s)\rangle \eta_\delta(Y_s-y)dyds \\ &\quad + \mathbb{E}\int_t^T\!\!\int_{\mathbb{R}^d} \langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle \eta_\delta(Y_s-y)dyd|K|_s\,, \end{aligned}$$

for $0 \le t \le T$, $\delta > 0$. Since $k_t \in \mathcal{C}$ and $\mathcal{C}$ is a closed convex cone, we have $\langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle \le 0$. Then, combining (2.4.19) with inequality $2d_c(.) \le 1 + d_c(.)^2$, we get

$$\begin{aligned} \mathbb{E}\phi_\delta(Y_t) \;&\le\; \delta^2 + C^0\mathbb{E}\int_t^T\!\!\int_{\mathbb{R}^d} |d_{\mathcal{C}}(y)|^2\eta_\delta(y-Y_s)dyds \\ &\quad + 2\mathbb{E}\int_t^T\!\!\int_{\mathbb{R}^d} d_{\mathcal{C}}(y)\eta_\delta(Y_s-y) \max_{y'\colon |y'-Y_s|\le\delta} |F(s,y',Z_s)-F(s,Y_s,Z_s)|dyds \\ &\le\; \delta^2 + C^0\int_t^T \mathbb{E}\phi_\delta(Y_s)ds + \mathbb{E}\int_t^T (1+\phi_\delta(Y_s)) \max_{y'\colon |y'-Y_s|\le\delta} |F(s,y',Z_s)-F(s,Y_s,Z_s)|ds\,. \end{aligned}$$

Using the Lipschitz property of F, we deduce

$$\mathbb{E}\phi_\delta(Y_t) \;\le\; C(\delta^2 + \delta + \int_t^T \mathbb{E}\phi_\delta(Y_s)ds),$$

and Gronwall's lemma leads to

$$\mathbb{E}\phi_\delta(Y_t) \;\le\; C(\delta^2+\delta), \quad 0 \le t \le T\,,\;\; \delta > 0\,.$$

Finally, from Fatou's Lemma, we have

$$\mathbb{E}|d_{\mathcal{C}}(Y_t)|^2 \;\le\; \liminf_{\delta\to 0} \mathbb{E}\phi_\delta(Y_t) \;=\; 0\,, \quad 0 \le t \le T\,,$$

which concludes the proof. □

2.4.2 A multi-dimentional comparison theorem for BSDEs

We now turn to the obtention of a multi-dimentional comparison result. Consider $(Y^1, Z^1, K) \in (\mathcal{S}^{c,2}_{\mathbb{F}} \times \mathbf{L}^2_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^2_{\mathbb{F}})^m$ satisfying

$$Y^1_t = Y^1_T + \int_t^T F_1(s, Y^1_s, Z^1_s)ds - \int_t^T \langle Z^1_s, dW_s\rangle + K_T - K_t$$

and $(Y^2, Z^2) \in (\mathcal{S}^{c,2}_{\mathbb{F}} \times \mathbf{L}^2_{\mathbb{F}}(\mathbf{W}))^m$ satisfying

$$Y^2_t = Y^2_T + \int_t^T F_2(s, Y^2_s, Z^2_s)ds - \int_t^T \langle Z^2_s, dW_s\rangle$$

Then we have the following comparison theorem generalizing the one in [43].

Theorem 2.4.2 *Suppose that $Y^1_T \geq Y^2_T$ and that, for any $(y, y', z, z') \in [\mathbb{R}^m]^2 \times [\mathbb{R}^{m\times d}]^2$, we have*

$$-4\langle y^-, F_1(t, y^+ + y', z) - F_2(t, y', z')\rangle \leq 2\sum_{i=1}^m \mathbf{1}_{y_i<0}|z_i - z'_i|^2 + 2C^0|y^-|^2 \ \mathbb{P} - a.s. \tag{2.4.21}$$

where $C^0 > 0$ is a constant. Then $Y^1_t \geq Y^2_t$, for all $t \in [0, T]$.

Proof. As in Theorem 2.1 in [43], it suffices to remark that F^1 is Lipschitz and apply Proposition 2.4.21 to the couple $(Y^1 - Y^2, Y^2)$ and the closed convex cone $\mathcal{C} = (\mathbb{R}^+)^m \times \mathbb{R}^m$. □

2.4.3 Monotonic Limit theorem for BSDE with jumps

This paragraph is devoted to the extension of Peng's monotonic limit theorem to the framework of BSDEs driven by a Brownian motion and a Poisson random measure.

Theorem 2.4.3 () *Let $(Y^n, Z^n, U^n, K^n)_n$ be a sequence in $\mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2_{\mathbb{G}}$ satisfying*

$$Y^n_t = Y^n_T + \int_t^T f(s, Y^n_s, Z^n_s, U^n_s)ds - \int_t^T \langle Z^n_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U^n_s(i)\mu(di, ds) + K^n_T - K^n_t,$$

for all $t \in [0, T]$. If $(Y^n)_n$ converges increasingly to Y with $\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^2] < \infty$, then $Y \in \mathcal{S}^2_{\mathbb{G}}$ (up to a modification) and there exists $(Z, U, K) \in \mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2_{\mathbb{G}}$ such that

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(di, ds) + K_T - K_t,$$

for all $t \in [0, T]$. Moreover (Z, U) is the weak (resp. strong) limit of the sequence $(Z^n, U^n)_n$ in $\mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ (resp. $\mathbf{L}^p_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^p(\tilde{\mu})$, for $p < 2$). Finally, K is the weak limit of $(K^n)_n$ in $\mathbf{L}^2_{\mathbb{G}}(\mathbf{0}, \mathbf{T})$ and, for any $t \in [0, T]$, K_t is the weak limit of $(K^n_t)_n$ in $\mathbf{L}^2(\mathbf{\Omega}, \mathcal{F}_\mathbf{t}, \mathbb{P})$.

Proof. The proof of Theorem 2.4.3 is an adaptation of the one presented in [64] and is performed in four steps.

1. *Uniform estimate.* Applying Itô's formula to $|Y^n|^2$ and using standard arguments (BDG inequality and Gronwall's Lemma) we get the existence of a constant $C > 0$ such that

$$\|Y^n\|_{\mathcal{S}^2_{\mathbb{G}}} + \|Z^n\|_{\mathbf{L}^2_{\mathbb{G}}(\mathbf{W})} + \|U^n\|_{\mathbf{L}^2(\tilde{\mu})} + \|K^n\|_{\mathcal{S}^2_{\mathbb{G}}} \leq C , \qquad n \in \mathbb{N} . \tag{2.4.22}$$

2. *Weak convergence.* Using the previous uniform estimate and the Hilbert structure of $\mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{0},\mathbf{T}) \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{0},\mathbf{T})$, we deduce the existence of a subsequence of $(Z^n, U^n, K^n, f(., Y^n, Z^n, U^n))_n$, which converges weakly to some (Z, U, K, F). Identifying the limits of $(Y^n)_n$ and $(Z^n, U^n, K^n, f(., Y^n, Z^n, U^n))_n$, we get

$$\begin{aligned} Y_t &= Y_T + \int_t^T F_s ds - \int_t^T \langle Z_s, dW_s \rangle \\ &\quad - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(di, ds) + K_T - K_t, \quad 0 \leq t \leq T. \end{aligned} \tag{2.4.23}$$

3. *Properties of the process K.* We first observe from Lemma 2.2 in [64] that the process K admits a càdlàg modification. We then establish that the contribution of the jumps of K is mainly concentrated within a finite number of intervals with sufficiently small total length. This result is derived with similar arguments as in [64], relying only the right continuity of the filtration and the predictability of the process K. As in Lemma 2.3 in [64], observe also that, for any $\delta, \epsilon > 0$, there exists a finite number of pairs of stopping times $(\sigma_k, \tau_k)_{0 \leq k \leq N}$ with $0 < \sigma_k \leq \tau_k \leq T$ such that

(i) $(\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset$ for $j \neq k$;

(ii) $\mathbb{E} \sum_{k=0}^N (\tau_k - \sigma_k) \geq T - \varepsilon$;

(iii) $\mathbb{E} \sum_{k=0}^N \sum_{\sigma_k < t \leq \tau_k} |\Delta K_t|^2 \leq \delta$.

4. *Strong convergence.* Following the arguments of the proof of Theorem 3.1 in [46], we deduce the convergence of $(Z^n, U^n)_n$ to (Z, U) in $dt \otimes d\mathbb{P}-$measure. Together with the uniform estimate (2.4.22), this leads to the strong convergence of $(Z^n, U^n)_n$ to (Z, U) in $\mathbf{L}^\mathbf{p}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^\mathbf{p}(\tilde{\mu})$, $p < 2$. Combining the Lipschitz property of f with (2.4.23), we conclude that (Y, Z, U, K) satisfies

$$\begin{aligned} Y_t &= Y_T + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T \langle Z_s, dW_s \rangle \\ &\quad - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(di, ds) + K_T - K_t, \quad 0 \leq t \leq T. \end{aligned}$$

□

Part II

DISCRETE-TIME APPROXIMATIONS OF BDSES ASSOCIATED WITH SEQUENTIAL OPTIMIZATION

Chapter 3

Probabilistic Representation and Approximation for Coupled Systems of Variational Inequalities

Abstract : This paper is dedicated to the probabilistic representation and approximation of solution to coupled systems of variational inequalities. The dynamics of each component of the solution is given by a different linear parabolic operator combined with a non linear dependence in all the components of the solution. This dynamics is coupled with a global structural constraint between all the components of the solution including the practical example of optimal switching problems. In this paper, we interpret the unique viscosity solution to this type of coupled systems of variational inequalities as the solution to one-dimensional constrained BSDEs with jumps introduced recently in [32]. In the spirit of [12], this new representation allows for the introduction of a natural entirely probabilistic numerical scheme for the resolution of these systems. We detail the algorithm and discuss its convergence.

Keywords: BSDE with jumps, variational inequalities, viscosity solutions, Monte Carlo simultations, Switching problems.

3.1 Introduction

The theory of stochastic differential equations provides a Feynman-Kac probabilistic representation for the solution of second order parabolic linear PDE's. Pardoux and Peng developed in [61] a theory for SDE with terminal conditions instead of initial one. These equations, called Backward Stochastic Differential Equations, provide a probabilistic representation of solution to quasilinear parabolic PDEs (see [62]). Coupling the diffusion process with a pure jump process, Pardoux Pradeilles and Rao [63] extend this representation to systems of coupled semilinear PDEs with different linear differential operators on each line. Introducing restrictions on the domain of the Backward process, El Karoui et all [29] cover the class of variational inequalities. Constraining instead the jump part of the Backward process, Kharroubi, Ma, Pham and Zhang [46] allow to consider quasilinear variational inequalities.

Our interest in this paper is to extend this type of Feynman-Kac representation to the more general class of coupled systems of quasilinear variational inequalities, arising for example in optimal impulse or switching problems. We will typically consider system of PDEs of the form

$$\Big[-\frac{\partial v_i}{\partial t} - \mathcal{L}^i v_i - f(i,.,(v_k)_{1\leq k\leq m}, \sigma^\top D_x v_i)\Big] \wedge \min_{1\leq j\leq m} h(i,.,v_i,v_j,\sigma^\top D_x v_i, j) = 0,$$
$$\text{on } [0,T)\times\mathbb{R}^d, \qquad \text{and} \qquad v_i(T,.) = g(i,.) \text{ on } \mathbb{R}^d, \qquad i\in\{1,\dots,m\} \tag{3.1.1}$$

where, for any $i\in\{1,\dots,m\}$, $\mathcal{L}^i$ is a linear second order local operator

$$\mathcal{L}^i v_i(t,x) \;=\; b(i,x).D_x v_i(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top(i,x)D_x^2 v_i(t,x))\,,$$

and b, σ, f and h are Lipschitz continuous functions. These equations appear in particular for the treatment of general switching problems in finite horizon, as observed in [78] or [54]. Consider an agent trying to maximize the outcomes of a running (and terminal) profit function f (and g), which are related to the path of a Brownian diffusion process X, whose dynamics depends on the current operating regime of production. At any time, the agent can switch between the different modes of productions, as long as the constraint given by h is satisfied. For example, if $h:(i,.,y_i,y_j,j)\mapsto y_i - y_j - c_{i,j}$, the difference between the value functions in both modes i and j stays above the constant $c_{i,j}$. This is the case whenever the agent has to pay a fixed cost $c_{i,j}$ in order to switch from regime i to regime j. The major difficulty for the study of this equation is due to the coupling between all the components v_i of the solution. In particular, when the number of modes m is high, the numerical resolution of (3.1.1) by classical PDE approximation methods is very tricky and highly computational. We intend to provide a probabilistic representation to (3.1.1) leading to an efficient numerical approximation scheme.

In the case where the drift μ and the volatility σ are independent on the regime of production (i.e. their first variable) and the constraint function is of the form $h:(i,.,y_i,y_j,j)\mapsto$

$y_i - y_j - c_{i,j}$, Hu and Tang [44] interpret the vector solution to (3.1.1) as a multi-dimensional BSDE with terminal condition and oblique reflections. To our knowledge, no numerical scheme is unfortunately available to approximate obliquely reflected BSDEs. The challenging derivation of a convergent numerical approximation for this type of BSDE is of great interest but is left for further research. The approach of this paper relies instead on a recent reinterpretation of obliquely multi-dimensional reflected BSDEs in terms of one-dimensional constrained BSDEs with Jumps, introduced by the authors in [32]. In the spirit of [63], the idea is to introduce a random mode of production given by a pure Jump process $(I_t)_t$ driven by an independant Poisson measure μ. Let consider a one-dimensional forward process X^I, whose dynamics are characterized by the random drift and volatility functions $\mu(., I_t)$ and $\sigma(., I_t)$, so that $\mathcal{L}^I$ is the Dynkin operator associated to $(X_t^I \mid I_t)$. Formally, given a smooth solution v to (3.1.1), the process $Y_t := v_{I_t}(t, X_t)$ satisfies

$$\begin{aligned} Y_t &= g(I_T, X_T) + \int_t^T f(I_s, X_s^I, Y_s + U_s, Z_s)ds + K_T - K_t \\ &\quad - \int_t^T Z_s.dW_s - \int_t^T \int_{\{1,\ldots,m\}} U_s(j)\mu(ds, dj), \end{aligned} \tag{3.1.2}$$

where $Z_t := \sigma^\top(I_{t^-}, X_t) D_x v_{I_{t^-}}(t, X_t)$, $U_t(.) := v_.(t, X_t) - v_{I_{t^-}}(t, X_t)$, and $K_t := \int_0^t (-\frac{\partial v_{I_t}}{\partial s} - \mathcal{L}^{I_s} v_{I_s} - f(., I_s))(s, X_s)ds$. Since v satisfies (3.1.1), we know that K is a continuous (hence predictable) nondecreasing process, and that the following constraint is satisfied:

$$h(I_{t^-}, X_t^I, Y_{t^-}, Y_{t^-} + U_t(j), Z_t, j) \quad \geq \quad 0, \quad j \in \{1, \ldots, m\}. \tag{3.1.3}$$

The Backward SDE (3.1.2) combined with the constraint (3.1.3) enters into the class of constrained BSDEs with jumps introduced in [32]. Under mild assumptions on the coefficients including the decreasing property of h in its fourth variable, there exists a unique minimal solution (Y, Z, U, K) to this BSDE. We prove in this paper that this solution interprets in terms of viscosity solution to the coupled system of variational inequalities (3.1.1). This Feynman-Kac representation extends the one in [46] since we consider general form of constraint functions h, and the dependence in U of the driver function implies a coupling between the different components of the solutions as in (3.1.1). Furthermore, the very general form of constraint function h enlarges also the conclusions of Peng and Xu [65] derived in the no-jump case. Finally, this result offers a Feynman-Kac representation for general reflected BSDEs with interconnected obstacles introduced recently by Hamadene and Zhang [41], and reinterpreted as constrained BSDEs with jumps in [32]. For the special case of optimal switching problems, our framework allows for the realistic consideration of cases where the dynamics of the forward process is influenced by the current switching regime.

Following the approach of [10] and [46], we provide under stronger assumptions a comparison theorem for the coupled system (3.1.1). Therefore, the component Y of the minimal solution to (3.1.2)-(3.1.3) interprets as the unique viscosity solution to (3.1.1). When the constraint function h does not depend on Z, the comparison result provides also an extra condition of the minimality for the constrained BSDE (3.1.2)-(3.1.3). This extra minimality condition is the following:

$$\int_0^T \min_{1\leq j\leq m} h(I_s, X_s^I, Y_{s-}, Y_{s-}+U_s, j)dK_s \;=\; 0\,. \tag{3.1.4}$$

The main interest of this new probabilistic representation for the solution to (3.1.1) is the obtention of a corresponding numerical algorithm. First, we approach the constrained BSDE with jumps by a sequence of penalized BSDEs with jumps. Second, we approximate the penalized BSDE using the algorithm introduced by Bouchard and Elie [12]. This leads to a converging numerical scheme based on time discretization, Monte Carlo simulations and numerical projections.

The rest of the paper is organized as follows: In Section 2, we precise the formulation of the constrained BSDEs with jumps considered here and the corresponding system of variational inequalities. We briefly recall existence and uniqueness results presented in [32], and we discuss the existence of a minimality condition. We interpret in Section 3 and 4 the minimal solution to the constrained BSDE as the unique viscosity solution to the corresponding system of parabolic variational inequalities. Finally, in Section 5 we discuss numerical issues and present the probabilistic algorithm.

Notations. Throughout this paper, we fix $m \in \mathbb{N}$ and denote $\mathcal{I} := \{1,\ldots,m\}$. Any element $x\in\mathbb{R}^d$ will be identified to a column vector with i-th component x^i and Euclidian norm $|x|$. For $x_i\in\mathbb{R}^{d_i}$, $i\leq n$ and $d_i\in\mathbb{N}$, we define $(x_1,\ldots,x_n)$ as the column vector associated to $(x_1^1,\ldots,x_1^{d_1},\ldots,x_n^1,\ldots,x_n^{d_n})$. For a $(m\times d)$-dimensional matrix M, we note $|M| := \sup\{|Mx|;\ x\in\mathbb{R}^d\ ,\ |x|=1\}$, $M^\top$ its transpose and we write $M\in\mathbb{M}^d$ if $m=d$. Given $p\in\mathbb{N}$ and a measured space $(A,\mathcal{A},\mu_A)$, we denote by $\mathbf{L}^p(A,\mathcal{A},\mu_A;\mathbb{R}^d)$, or simply $\mathbf{L}^p(A,\mathcal{A})$ or $\mathbf{L}^p(A)$ if no confusion is possible, the set of p-integrable $\mathbb{R}^d$-valued measurable maps on $(A,\mathcal{A},\mu_A)$. For $p=\infty$, $\mathbf{L}^\infty(A,\mathcal{A},\mu_A;\mathbb{R}^d)$ is the set of essentially bounded $\mathbb{R}^d$-valued measurable maps. For a function $v:\ [0,T]\times\mathbb{R}^d\to\mathbb{R}^m$ and $(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathcal{I}$ we denote by $v(t,x,i)$ the i-th component of the vector $v(t,x)\in\mathbb{R}^m$. $C^{1,2}([0,T]\times\mathbb{R}^d\times\mathcal{I})$ (resp. $C^2(\mathbb{R}^d\times\mathcal{I})$) denotes the set of functions $\varphi:\ [0,T]\times\mathbb{R}^d\times\mathcal{I}\to\mathbb{R}$ (resp. $\varphi:\ \mathbb{R}^d\times\mathcal{I}\to\mathbb{R}$) such that $\varphi(.,i)\in C^{1,2}([0,T]\times\mathbb{R})$ (resp. $\varphi(.,i)\in C^2(\mathbb{R}^d)$), for all $i\in\mathcal{I}$. For a smooth function $\varphi:\ [0,T]\times\mathbb{R}^d\times\mathcal{I}\to\mathbb{R}$, $\frac{\partial\varphi}{\partial t}$, $D_x\varphi$ and $D_x^2\varphi$ denote respectively the derivative of φ w.r.t. t, the gradient and the Hessian matrix of φ w.r.t. x. For ease of notation, we omit in all the paper the dependence in $\omega\in\Omega$, whenever it is explicit.

3.2 Constrained Forward Backward SDEs with jumps

We present in this section the notion of constrained Forward Backward SDEs with jumps and briefly recall the existence and uniqueness results derived in [32]. We detail the approximation procedure based on penalization, which will prove its interest in the following sections. We discuss the correspondence between the value function associated to Y and all the components of the solution. Under an extra regularity assumption on the value function, we provide a Skorohod type minimality condition for the considered BSDE.

3.2.1 Formulation

Throughout this paper we are given a finite terminal time T and a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$, and an independent Poisson random measure μ on $\mathbb{R}_+ \times \mathcal{I}$, where $\mathcal{I} = \{1, \ldots, m\}$, with intensity measure $\lambda(di)dt$ for some finite measure λ on $\mathcal{I}$ with $\lambda(i) > 0$, for all $i \in \mathcal{I}$. We set $\tilde{\mu}(dt, di) = \mu(dt, di) - \lambda(di)dt$ the compensated measure associated to μ. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ (resp.$\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$) the augmentation of the natural filtration generated by W and μ (resp. by W), and by $\mathcal{P}$ the σ-algebra of predictable subsets of $\Omega \times [0, T]$.

The Forward process of the equation is composed of both a pure jump process I and a diffusion X without jump depending on I. Let $b : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be continuous functions, Lipschitz in their second variable uniformly in their first variable. For each initial condition $(t, i, x) \in [0, T] \times \mathcal{I} \times \mathbb{R}^d$, let $(I_s^{t,i}, X_s^{t,i,x})_{t\leq s\leq T}$ be the unique $\mathcal{I} \times \mathbb{R}^d$-valued solution of the SDE:

$$\left\{ \begin{array}{lcl} I_s &=& i + \int_t^s \int_{\mathcal{I}} (j - I_{r^-})\mu(dr, dj) \\ X_s &=& x + \int_t^s b(I_r, X_r)dr + \int_t^s \sigma(I_r, X_r)dW_r \end{array} \right. . \tag{3.2.1}$$

Before introducing the backward SDE, we need to define some additional notations. For any $p > 2$, We denote by $\mathcal{S}^p$ the set of real valued $\mathcal{G}$-adapted *càdlàg* processes Y on $[0, T]$ such that

$$\|Y\|_{\mathcal{S}^p} \ := \ \mathbb{E}\left[\sup_{0\leq r\leq T} |Y_r|^p\right]^{\frac{1}{p}} \ < \ \infty \,,$$

$\mathbf{L}_W^p$ is the set of progressively measurable $\mathbb{R}^d$-valued processes Z such that

$$\|Z\|_{\mathbf{L}_W^p} \ := \ \mathbb{E}\left[\left(\int_0^T |Z_r|^p dr\right)\right]^{\frac{1}{p}} \ < \ \infty \,,$$

$\mathbf{L}_{\tilde{\mu}}^p$ is the set of $\mathcal{P} \otimes \sigma(\mathcal{I})$ measurable maps $U : \Omega \times [0, T] \times \mathcal{I} \to \mathbb{R}$ such that

$$\|U\|_{\mathbf{L}_{\tilde{\mu}}^p} \ := \ \mathbb{E}\left[\int_0^T \int_{\mathcal{I}} |U_s(j)|^p \lambda(dj)ds\right]^{\frac{1}{p}} \ < \ \infty \,,$$

(here $\sigma(\mathcal{I})$ denotes the class of subsets of $\mathcal{I}$) and $\mathbf{A}^2$ is the closed subset of $\mathcal{S}^2$ composed by nondecreasing processes K with $K_0 = 0$.

For any initial condition $(t, i, x) \in [0, T] \times \mathcal{I} \times \mathbb{R}^d$, we consider the constrained BSDE with jumps, whose dynamics is given by

$$\begin{aligned} Y_t &= g(I_T^{t,i}, X_T^{t,i,x}) + \int_t^T f(I_s^{t,i}, X_s^{t,i,x}, Y_s + U_s, Z_s)ds + K_T - K_t \\ &\quad - \int_t^T Z_s.dW_s - \int_t^T \int_{\mathcal{I}} U_s(j)\mu(ds, dj), \end{aligned} \tag{3.2.2}$$

together with the constraint

$$h(I_{t-}^{t,i}, X_t^{t,i,x}, Y_{t-}, Y_{t-} + U_t(j), Z_t, j) \;\; \geq \;\; 0, \quad j \in \mathcal{I}\,, \tag{3.2.3}$$

where g, f and h are deterministic functions.

A solution to the constrained BSDE with jumps is a quadruplet $(Y, Z, U, K) \in \mathcal{S}^2 \times \mathbf{L}^2_W \times \mathbf{L}^2_{\tilde{\mu}} \times \mathbf{A}^2$ satisfying (3.2.2)-(3.2.3). Furthermore, (Y, Z, U, K) is referred to as the minimal solution to (3.2.2)-(3.2.3) whenever, for any other solution (Y', Z', U', K') to (3.2.2)-(3.2.3), we have $Y \leq Y'$ a.s..

3.2.2 Existence and uniqueness of a minimal solution

In order to ensure existence and uniqueness to the constrained BSDE with jumps (3.2.2)-(3.2.3) starting from any initial condition $(t, i, x) \in [0, T] \times \mathcal{I} \times \mathbb{R}^d$, we need to adapt the assumptions presented in [32] and introduce:

(H0)

(i) There exists a constant $k > 0$ such that the functions f and h satisfy the uniform Lipschitz property

$$\begin{aligned} |f(i, x, y + (u_j)_{j\in\mathcal{I}}, z) - f(i, x, y' + (u'_j)_{j\in\mathcal{I}}, z')| &\leq k|(y, z, (u_j)_{j\in\mathcal{I}}) - (y', z', (u'_j)_{j\in\mathcal{I}})| \\ |h(i, x, y, y + u_j, z, j) - h(i, x, y', y' + u'_j, z', j)| &\leq k|(y, z, u_j) - (y', z', u'_j)|\,, \end{aligned}$$

for all $(x, i, j, y, z, u, y', z', u') \in \mathbb{R}^d \times \mathcal{I}^2 \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}]^2$.

(ii) The coefficients $f(i, .)$, $g(i, .)$ and $h(i, ., j)$ are continuous for any $i, j \in \mathcal{I}$ and they satisfy the following growth linear condition : there exists a constant C such that

$$|f(i, x, y + u, z)| + |g(i, x)| + |h(i, x, y, y + u_j, z, j)| \;\; \leq \;\; C(1 + |x| + |y| + |z| + |u|)\,,$$

for all $(x, i, j, y, z, u) \in \mathbb{R}^d \times \mathcal{I}^2 \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}$.

(iii) There exist two constants $C_1 \geq C_2 > -1$ and a measurable map $\gamma : \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^{\mathcal{I}}]^2 \times \mathcal{I} \to \mathbb{R}$ such that $C_2 \leq \gamma(.) \leq C_1$ and

$$f(i,x,y+u,z) - f(i,x,y'+u',z') \quad \leq \quad \int_{\mathcal{I}} (u_j - u'_j)\gamma(i,x,y,z,u,u',j)\lambda(dj)\,,$$

for all $(i,x,y,z,u,u') \in \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^{\mathcal{I}}]^2$.

(iv) The function $h(i,x,y,y+.,z,j)$ is decreasing for all $(i,x,y,z,j) \in \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{I}$.

Assumptions (i) and (ii) are classical for the study of BSDEs and their links with PDEs. (iii) and (iv) allows to set comarison theorem for penalized BSDEs defined below and used to construct the minimal constrained solution (see [32]).

In order to ensure that this problem is well defined, we also need to assume:

(H1) For any initial condition, there exists a quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U}) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L}^2_W \times \mathbf{L}^2_{\tilde{\mu}} \times \mathbf{A}^2$ satisfying (3.2.2)-(3.2.3), with $\tilde{Y}_t = \tilde{v}(t, X_t, I_t)$, $\leq t \leq T$, for some deterministic function $\tilde{v}$ satisfying a linear growth condition

$$|\tilde{v}(t,x,i)| \quad \leq \quad C(1+|x|)\,, \quad (t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}\,.$$

Remark 3.2.1 Being aware that Assumption **(H1)** is rather restrictive, we provide in Remark 3.3.2 a more tractable sufficient condition for it. Notice that **(H1)** is slightly stronger than its analogous in [32]. This allows to ensure the corresponding Markovian value function defined below to satisfy a linear growth condition and to be in particular locally bounded, see Lemma 3.3.1.

As a direct application of Theorem 2.1 in [32], we verify the following.

Theorem 3.2.1 *Suppose Assumptions* **(H0)** *and* **(H1)** *hold. For all triple* $(t,i,x) \in [0,T] \times \mathcal{I} \times \mathbb{R}^d$*, there exists a unique quadruple* $(Y^{t,i,x}, Z^{t,i,x}, U^{t,i,x}, K^{t,i,x}) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L}^2_W \times \mathbf{L}^2_{\tilde{\mu}} \times \mathbf{A}^2$ *minimal solution to* (3.2.2)-(3.2.3)*, and* $v(t,x,i) := Y^{t,i,x}_t$ *defines a deterministic map from* $[0,T] \times \mathbb{R}^d \times \mathcal{I}$ *into* $\mathbb{R}$.

3.2.3 Related penalized BSDE

In [32], the proof for the existence of a minimal solution to (3.2.2)-(3.2.3) relies on a penalization argument. We will require the corresponding penalized BSDE in the next sections and we choose to present it here. For any initial condition $(t,i,x) \in [0,T] \times \mathcal{I} \times \mathbb{R}^d$ and $n \in \mathbb{N}$, we denote by $(Y^{t,i,x,n}, Z^{t,i,x,n}, U^{t,i,x,n})$ the solution to the following penalized BSDE with jump

$$\begin{aligned} Y_t \;=\;& g(I^{t,i}_T, X^{t,i,x}_T) + \int_t^T f(I^{t,i}_s, X^{t,i,x}_s, Y_s + U_s, Z_s)ds - \int_t^T \int_{\mathcal{I}} U_s(j)\mu(ds,dj) \qquad (3.2.4)\\ -\;& \int_t^T Z_s.dW_s + n\int_t^T \int_{\mathcal{I}} h^-(I^{t,i}_{s^-}, X^{t,i,x}_s, Y_{s^-}, Y_{s^-} + U_s(j), Z_s, j)\lambda(dj)ds\,, \quad 0 \leq t \leq T\,. \end{aligned}$$

Under **(H0)**, following the arguments of [5], we verify that there exists a unique solution to (3.2.4), for any $n \in \mathbb{N}$. From [32] we have the following result:

Theorem 3.2.2 *Under* **(H0)**-**(H1)***, for any* $(t,i,x) \in [0,T]\times\mathcal{I}\times\mathbb{R}^d$*, the sequence* $(Y^{t,i,x,n})_n$ *is nondecreasing and converges to* $Y^{t,i,x}$ *in the folowing sense:*

$$\mathbb{E}\Big[\int_t^T \Big|Y_s^{t,i,x,n} - Y_s^{t,i,x}\Big|^2 ds\Big] + \mathbb{E}\Big[\Big|Y_\tau^{t,i,x,n} - Y_\tau^{t,i,x}\Big|\Big] \underset{n\to\infty}{\longrightarrow} 0,$$

for any stopping time τ *valued in* $[\tau, T]$*. Moreover we have*

$$\|Z^{t,i,x} - Z^{t,i,x,n}\|_{\mathbf{L}^p_W} + \|U^{t,i,x} - U^{t,i,x,n}\|_{\mathbf{L}^p_{\tilde{\mu}}} \underset{n\to\infty}{\longrightarrow} 0\,, \qquad p<2\,. \tag{3.2.5}$$

Under an extra regularity property of the process $Y^\cdot$, we can improve this convergence result.

Proposition 3.2.1 *If* **(H0)**-**(H1)** *holds and the process* $Y^\cdot$ *is quasi-left continuous in time, we have*

$$\|Y^\cdot - Y^{\cdot,n}\|_{\mathcal{S}^2} + \|Z^\cdot - Z^{\cdot,n}\|_{\mathbf{L}^2_W} + \|U^\cdot - U^{\cdot,n}\|_{\mathbf{L}^2_{\tilde{\mu}}} + \|K^\cdot - K^{\cdot,n}\|_{\mathcal{S}^2} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{3.2.6}$$

Proof. First notice that, according to [63], the process $Y^{\cdot,n}$ is also quasi-left continuous. Therefore, the predictable projections of $Y^\cdot$ and $Y^{\cdot,n}$ are simply given by $(Y_{t^-}{}^\cdot)_t$ and $(Y_{t^-}^{\cdot,n})_t$ and satisfy $Y_{\cdot t^-} = \lim_{n\to\infty} Y_{t^-}^{\cdot,n}$. We deduce from the weak version of Dini's theorem, see [26] p. 202, that $Y^{\cdot,n}$ converges uniformly to $Y^\cdot$ on $[0,T]$, and the dominated convergence theorem leads to $\|Y^\cdot - Y^{\cdot,n}\|_{\mathcal{S}^2} \underset{n\to\infty}{\longrightarrow} 0$. Combined with standard estimates of the form

$$\|Z^{n+p} - Z^n\|^2_{\mathbf{L}^2_W} + \|U^{n+p} - U^n\|^2_{\mathbf{L}^2_{\tilde{\mu}}} + \|K^{n+p} - K^n\|^2_{\mathcal{S}^2} \;\leq\; C\|Y^{n+p} - Y^n\|^2_{\mathcal{S}^2}\,.$$

this implies that the sequences (Z^n), (U^n) and (K^n) are Cauchy in their respective Banach spaces, which concludes the proof. □

Remark 3.2.2 Under the extra Assumptions **(H2)** and **(H3)** below, the value function $v : (t,x,i) \mapsto Y_t^{t,x,i}$ interprets as the unique viscosity solution to a well chosen system of variational inequalities. In this case, v is continuous and $Y_t = v(t, X_t, I_t)$ is quasi left continuous so that we derive the strong convergence of the penalized BSDE to the constrained one.

In order to derive viscosity properties on the solution to the constrained BSDE with jump (3.2.2)-(3.2.3), we shall pass to the limit the viscosity properties of (3.2.4). Similarly in Section 3.5, in order to approximate numerically the solution to (3.2.2)-(3.2.3), we shall work directly on the penalized one.

3.2.4 Link between (Y, U) and X

We give here a representation of the processes Y and U as deterministic functional of the process X.

Let define the sequence of functions $v_n(resp.\ v) : [0,T] \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$ by

$$v_n(resp.\ v)\ :\ (t,x,i) \quad \mapsto \quad Y_t^{t,i,x,n}\ (resp.\ Y_t^{t,i,x}), \quad (t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}, \tag{3.2.7}$$

where, for any initial condition $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathcal{I}$ $(Y_s^{t,i,x,n}, Z_s^{t,i,x,n}, U_s^{t,i,x,n})_{t\leq s\leq T}$ is the minimal solution to BSDE (3.2.2)-(3.2.3). Let first observe the following identification between v_n and the components of the solution to the BSDE (3.2.4).

Lemma 3.2.1 *Let Assumption* **(H0)-(H1)** *hold. For all $(t,i,j,x) \in [0,T]^2 \times \mathcal{I}^2 \times \mathbb{R}^d$, we have the following identifications:*

$$Y_\theta^{t,i,x,n} \quad = \quad v_n(\theta, X_\theta^{t,i,x}, I_\theta^{t,i}) \tag{3.2.8}$$

for any stopping time θ valued in $[t,T]$, and

$$U_s^{t,i,x,n}(j) \quad = \quad v_n(s, X_s^{t,i,x}, j) - v_n(s, X_s^{t,i,x}, I_{s^-}^{t,i})\,, \tag{3.2.9}$$

for any $s \in [t,T]$.

Proof. Fix $(t,i,j,x) \in [0,T] \times \mathcal{I}^2 \times \mathbb{R}^d$ and θ a stopping time valued in $[t,T]$. The identification of $Y^{\cdot,n}$ comes from uniqueness of solution to (3.2.4) and from the Markov property of (I,X). Let us now check the identification of $U^{\cdot,n}$ and define

$$\tilde{U}_s^{t,i,x,n}(j) \quad := \quad v_n(s, X_s^{t,i,x}, j) - v_n(s, X_s^{t,i,x}, I_{s^-}^{t,i}).$$

Using (3.2.4) and the identification of $Y^{\cdot,n}$, we derive

$$\int_{\mathcal{I}} U_s^{t,i,x,n}(j)\mu(ds,dj) \quad = \quad Y_s^{t,i,x,n} - Y_{s-}^{t,i,x,n} \quad = \quad \int_{\mathcal{I}} \tilde{U}_s^{t,i,x,n}(j)\mu(ds,dj)\,.$$

which gives $\int_{\mathcal{I}}(U_s^{t,i,x,n}(j) - \tilde{U}_s^{t,i,x,n}(j))^2\mu(ds,dj) = \Big[\int_{\mathcal{I}}(U_s^{t,i,x,n}(j) - \tilde{U}_s^{t,i,x,n}(j))\mu(ds,dj)\Big]^2 = 0$. Hence, $\mathbf{E}[\int_0^T \int_{\mathcal{I}}(U_s^{t,i,x,n}(j) - \tilde{U}_s^{t,i,x,n}(j))^2\mu(ds,dj)] = 0$ which implies that

$$\mathbf{E}\left[\int_0^T \int_{\mathcal{I}}(U_s^{t,i,x,n}(j) - U_s^{t,i,x,n}(j))^2\lambda(dj)ds\right] = 0\,,$$

and concludes the proof. □

Proposition 3.2.2 *Let Assumption* **(H0)-(H1)** *hold. The function v links the processes $Y^{t,i,x}$ and $U^{t,i,x}$ with $X^{t,i,x}$ by the relation:*

$$Y_\theta^{t,i,x} \quad = \quad v(\theta, X_\theta^{t,i,x}, I_\theta^{t,i,x}), \tag{3.2.10}$$

for any stopping time θ valued in $[t,T]$, and

$$U_s^{t,i,x}(j) \quad = \quad v(s, X_s^{t,i,x}, j) - v(s, X_s^{t,i,x}, I_{s^-}^{t,i})\,, \tag{3.2.11}$$

for any $s \in [t,T]$.

Proof. From Theorem 3.2.2 and the definition of v, we know that v is the pointwise limit of $(v_n)_n$. We deduce (3.2.10) from (3.2.8) and (3.2.11) is implied by (3.2.9) and (3.2.5). □

3.2.5 The minimality condition

Under an extra regularity assumption on the function v satisfied under Assumptions **(H2)** and **(H3)** below, we deduce from the previous proposition a minimality condition for the solution (Y, Z, U, K), which is a Skorohod type condition as in [29] in the case of reflected BSDEs.

Corollary 3.2.1 *Let Assumption* **(H0)-(H1)** *hold. Suppose that the function $v(., i)$ is continuous on $[0, T) \times \mathbb{R}^d$ for all $i \in \mathcal{I}$, and that the function h does not depend on the component Z. Then for all $(t, i, x) \in [0, T] \times \mathcal{I} \times \mathbb{R}^d$ the minimal solution $(Y^{t,i,x}, Z^{t,i,x}, U^{t,i,x}, K^{t,i,x})$ satisfy the Skorohod condition*

$$\int_t^T \min_{j \in \mathcal{I}} \Big[h(I_{s^-}^{t,i,x}, Y_{s^-}^{t,i,x}, U_s^{t,i,x}(j), j) \Big] dK_s^{t,i,x} \quad = \quad 0 \tag{3.2.12}$$

Proof. Notice first that if $v(., i)$ is continuous for all $i \in \mathcal{I}$, then the processe $Y^{t,i,x}$ is quasi-left continuous since $(I^{t,i,x}, X^{t,i,x})$ is too. From Proposition 3.2.1 and the representation (3.2.11), we have also

$$\|Y^{t,i,x} - Y^{t,i,x,n}\|_{\mathcal{S}^2} + \max_{j \in \mathcal{I}} \|U^{t,i,x}(j) - U^{t,i,x,n}(j)\|_{\mathcal{S}^2} \underset{n \to \infty}{\longrightarrow} 0\,. \tag{3.2.13}$$

In particular we get (see Lemma 5.8 in [35] which is also true for càglàd functions)

$$\begin{aligned} \int_t^T \min_{j \in \mathcal{I}} \Big[h(I_{s^-}^{t,i,x}, X_s^{t,i,x}, Y_{s^-}^{t,i,x,n}, U_s^{t,i,x,n}(j), j) \Big] dK_s^{t,i,x,n} & \underset{n \to \infty}{\longrightarrow} \\ \int_t^T \min_{j \in \mathcal{I}} \Big[h(I_{s^-}^{t,i,x}, Y_{s^-}^{t,i,x}, U_s^{t,i,x}(j), j) \Big] dK_s^{t,i,x} & \end{aligned} \tag{3.2.14}$$

Then using

$$\int_t^T \min_{j \in \mathcal{I}} \Big[h(I_{s^-}^{t,i,x}, X_s^{t,i,x}, Y_{s^-}^{t,i,x,n}, U_s^{t,i,x,n}(j), j) \Big] dK_s^{t,i,x,n} \quad \leq \quad 0,$$

and

$$\int_t^T \min_{j \in \mathcal{I}} \Big[h(I_{s^-}^{t,i,x}, Y_{s^-}^{t,i,x}, U_s^{t,i,x}(j), j) \Big] dK_s^{t,i,x} \quad \geq \quad 0,$$

we get (3.2.12). □

3.3 Link with coupled systems of variational inequalities

This section is devoted to the viscosity property of the minimal solution of the constrained BSDE with Jump (3.2.2)-(3.2.3). Generalizing the representation derived in [46] and [65], we interpret the value function v as a viscosity solution to a system of coupled variational inequalities given below by (3.3.2)-(3.3.3). The argument relies on a passing to the limit the viscosity properties of the corresponding penalized BSDEs with jumps.

3.3.1 Viscosity properties of the penalized BSDE

The penalized parabolic integral partial differential equation (IPDE) associated to (3.2.4) is naturally defined, for each $n \in \mathbb{N}$, by

$$\left\{ \begin{array}{lll} -\dfrac{\partial \varphi}{\partial t}(.,i) - \mathcal{L}^i\varphi(.,i) - f(i,x,(\varphi(.,j))_{j\in\mathcal{I}}, \sigma^\top D_x\varphi(.,i)) & & \\ -n \int_{\mathcal{I}} [h(i,x,\varphi(.,i)\varphi(.,j), \sigma^\top D_x\varphi(.,i),j)]^- \lambda(dj) = 0 & \text{on} & [0,T) \times \mathbb{R}^d \times \mathcal{I}, \\ v(T,x,i) = g(x,i) & \text{on} & \mathbb{R}^d \times \mathcal{I}\,, \end{array} \right. \tag{3.3.1}$$

where $\mathcal{L}$ is the m-dimensional second order local operator associated to X and given by

$$\mathcal{L}^i\varphi \;\; := \;\; b^\top(i,.)D_x\varphi + \frac{1}{2}\mathrm{tr}[\sigma(i,.)\sigma(i,.)^\top D_x^2\varphi]\,, \qquad \text{on } [0,T] \times \mathbb{R}^d \times \mathcal{I}\,.$$

Since the penalized BSDE enters into the class of BSDE with jumps studied by Pardoux, Pradeilles and Rao [63], we have the following Feynman-Kac representation result.

Proposition 3.3.1 *Under* **(H0)-(H1)** *and for any $n \in \mathbb{N}$, the functions v_n defined in* (3.2.7) *are continuous viscosity solutions to* (3.3.1). *Indeed, for any $n \in \mathbb{N}$ and $i \in \mathcal{I}$, $v_n(T,.,i) = g(.,i)$ and, for any $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d \times \mathcal{I})$ such that (t,x) is a global minimum (resp. maximum) of $(v_n - \varphi)(.,i)$ for all $i \in \mathcal{I}$, we have*

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(.,i) - \mathcal{L}^i\varphi(.,i) - f(i,.,(\varphi(.,j))_{j\in\mathcal{I}}, \sigma^\top D_x\varphi(.,i)) & \\ -n\int_{\mathcal{I}} h^-(.,\varphi(.,i),\varphi(.,j),\sigma^\top D_x\varphi(.,i),j)\lambda(dj) \;\; & \geq (\text{ resp. } \leq) \;\; 0\,, \;\; i \in \mathcal{I}\,. \end{aligned}$$

Proof. The continuity of v_n follows from the same argument as in the proof of Lemma 2.1 in [63]. Similarly, according to Lemma 3.2.1, the viscosity property of v_n exactly fits in the framework of Theorem 4.1 in [63]. □

3.3.2 Viscosity properties of the constrained BSDE with jumps

We state in this subsection the viscosity property of the function v. Formally, passing to the limit in (3.3.1) when n goes to infinity, we expect v to be solution, on $[0,T) \times \mathbb{R}^d \times \mathcal{I}$, to the following coupled system of variational inequalities

$$\begin{aligned} \min\Big[-\frac{\partial v}{\partial t}(.,i) - \mathcal{L}^i v(.,i) - f\left(i,.,(v(.,j))_{j\in\mathcal{I}}, \sigma^\top D_x v(.,i)\right)\,, & \\ \min_{j\in\mathcal{I}} h\left(i,.,v(.,i),v(.,j),\sigma^\top D_x v(.,i),j\right)\Big] \;\; & = \;\; 0, \;\; i \in \mathcal{I}\,. \end{aligned} \tag{3.3.2}$$

To complete the PDE characterization of the function v, we need to provide a suitable boundary condition. In general, we can not expect to have $v(T^-,.) = g$, and we shall consider the relaxed boundary condition given by

$$\min\left[v(T^-,x,i) - g(x,i), \min_{j\in\mathcal{I}} h\left(i,x,v(.,i),v(.,j),\sigma^\top D_x v(.,i),j\right)\right] = 0 \text{ on } \mathbb{R}^d \times \mathcal{I}. \tag{3.3.3}$$

Remark 3.3.1 In the particular case where the driver function f is independent of (y, z) and the constraint function is given by $\tilde{h} : (i, x, y, y + v, z, j) \mapsto -c(i, j) - v$ with c a given cost function, the variational inequality (3.3.2) rewrites equivalently as

$$\min \Big[-\frac{\partial v}{\partial t}(., i) - \mathcal{L}^i v(., i) - f(i, .),\ \min_{j \in \mathcal{I}} [v(., i) - v(., j) - c(i, j)] \Big] \quad = \quad 0,\ i \in \mathcal{I},$$

so that we retrieve the classical variational inequalities associated to switching problems. Furthermore, the corresponding relaxed boundary condition (3.3.3) rewrites as

$$\min \Big[v(T^-, ., i) - g(., i),\ \min_{j \in \mathcal{I}} [v(T^-, ., i) - v(T^-, ., j) - c(i, j)] \Big] \quad = \quad 0, \text{ on } \mathbb{R}^d \times \mathcal{I}. \tag{3.3.4}$$

Therefore, if (3.3.4) satisfies a comparison theorem, $v(T^-, .)$ interprets as the smallest function grower to g satisfying (3.3.4) (see remark 3.4 in [66]). In particular, we retrieve the terminal condition $v(T^-, .) = g$ proposed by [44] in the framework of reflected BSDE, since their terminal condition g is assumed to satisfy the constraint.

Since we consider solutions in a discontinuous viscosity sense, we introduce, for any locally bounded function u on $[0, T] \times \mathbb{R}^d \times \mathcal{I}$, its lower semicontinuous and upper semicontinuous (lsc and usc in short) envelopes u_* and u^* defined by

$$u_*(t, x, i) \;=\; \liminf_{(t', x') \to (t, x), t' < T} u(t', x', i), \qquad u^*(t, x, i) \;=\; \limsup_{(t', x') \to (t, x), t' < T} u(t', x', i)\,,$$

for $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$.

We now turn to the definition of viscosity solutions to (3.3.2)-(3.3.3).

Definition 3.3.1 *(Viscosity solutions to* (3.3.2)-(3.3.3)*)*
(i) A function u, lsc (resp. usc) on $[0, T) \times \mathbb{R}^d \times \mathcal{I}$, is called a viscosity supersolution (resp. subsolution) to (3.3.2)-(3.3.3) *if, for each $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathcal{I}$ and any $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d \times \mathcal{I})$ such that (t, x) is a global minimum (resp. maximum) of $(u - \varphi)(., i)$ for all $i \in \mathcal{I}$, we have, if $t < T$,*

$$\begin{aligned} \min \Big[-\frac{\partial \varphi}{\partial t}(., i) - \mathcal{L}^i \varphi(., i) - f(i, ., (\varphi(., j))_{j \in \mathcal{I}}, \sigma^\top D_x \varphi(., i)), \\ \min_{j \in \mathcal{I}} h(i, x, \varphi(., i), \varphi(., j), \sigma^\top D_x \varphi(., i), j) \Big](t, x) \quad \geq (\textit{ resp. } \leq)\ \ 0\,,\ \ i \in \mathcal{I}\,, \end{aligned}$$

and, if $t = T$,

$$\min \Big[u(T, x, i) - g(x, i),\ \min_{j \in \mathcal{I}} h(i, x, \varphi(., i), u(., j), \sigma^\top D_x \varphi(., i), j) \Big] \quad \geq (\textit{ resp. } \leq)\ \ 0\,,\ \ i \in \mathcal{I}.$$

(ii) A locally bounded function u on $[0, T) \times \mathbb{R}^d \times \mathcal{I}$ is called a viscosity solution to (3.3.2)-(3.3.3) *if u_* and u^* are respectively viscosity supersolution and subsolution to* (3.3.2)-(3.3.3)*.*

We first observe that v is locally bounded.

Lemma 3.3.1 *Under* **(H0)** *and* **(H1)***, the function v satisfies the linear growth property*

$$\sup_{t\in[0,T]} |v(t,x,i)|^2 \;\leq\; C(1+|x|^2)\,, \quad i\in\mathcal{I}\,. \tag{3.3.5}$$

Proof. Following the lines of the proof of Lemma 3.3 and Remark 3.2 in [46], standard estimates on the penalized BSDE (3.2.4) lead to

$$\sup_{t\in[0,T]} |Y_t^{t,i,x,n}|^2 \leq C\Big(1+\mathbb{E}\Big[|g(X_T^{t,i,x},I_T^{t,i})|^2+\int_t^T |X_s^{t,i,x}|^2ds+\sup_{s\in[0,T]}|\tilde{v}(s,X_s^{t,i,x},I_s^{t,i})|^2\Big]\Big),$$

for any $i\in\mathcal{I}$. Combining Fatou's lemma with the standard estimates on X and the linear growth condition on g and $\tilde{v}$, see **(H1)**, we directly compute (3.3.5). □

In order to derive the viscosity properties of the value function v, we shall appeal to the following dynamic programming characterization of the minimal solution.

Lemma 3.3.2 *Let $(t,i,x)\in[0,T)\times\mathcal{I}\times\mathbb{R}^d$, and $(Y^{t,i,x},Z^{t,i,x},U^{t,i,x},K^{t,i,x})$ be the minimal solution to* (3.2.2)-(3.2.3) *on $[t,T]$ with $(X_s,I_s)=(X_s^{t,i,x},I_s^{t,i})$. Then, for any stopping time θ valued in $[t,T]$, $(Y_s^{t,i,x},Z_s^{t,i,x},U_s^{t,i,x},K_s^{t,i,x})_{s\in[t,\theta]}$ is a minimal solution to :*

$$\begin{aligned} Y_s &= v(\theta,X_\theta^{t,i,x},I_\theta^{t,i})+\int_s^\theta f(I_r^{t,i},X_r^{t,i,x},Y_r+U_r,Z_r)dr+K_\theta-K_s \\ &\quad -\int_s^\theta Z_r.dW_r-\int_s^\theta\int_{\mathcal{I}}U_r(j)\mu(dr,dj), \quad t\leq s\leq\theta, \ a.s. \end{aligned} \tag{3.3.6}$$

with

$$h(I_{r^-}^{t,i},X_r^{t,i,x},Y_{r^-},Y_{r^-}+U_r(j),Z_r,j)\geq 0 \quad d\mathbb{P}\otimes dt\otimes\lambda(dj) \quad a.e.\ on\ \Omega\times[t,\theta]\times\mathcal{I}. \tag{3.3.7}$$

Proof. We omit the standard proof of this lemma which is based on the same probabilistic arguments as the proof of lemma 4.1 in [46]. It classically relies on a concatenation of solutions to the BSDE on the time intervals $[t,\theta(\omega)]$ and $[\theta(\omega),T]$. □

We now turn to the main result of the section.

Theorem 3.3.1 *Under* **(H0)-(H1)***, the function v is a (discontinuous) viscosity solution to* (3.3.2)-(3.3.3).

Proof of Theorem 3.3.1. First notice that v is locally bounded, according to lemma 3.3.1. • *Viscosity property on $[0,T)\times\mathbb{R}^d$.*

From the results of the previous section, we know that v is the pointwise limit of the nondecreasing sequence of functions (v_n). By continuity of v_n, we then have (see e.g. [4] p. 91) :

$$v \;=\; v_* \;=\; \lim_{n\to\infty}\inf{}_* v_n, \text{ where } \lim_{n\to\infty}\inf{}_* v_n(t,x,i) := \liminf_{\substack{n\,\to\,\infty \\ t'\,\to\,t,\,x'\,\to\,x}} v_n(t',x',i), \tag{3.3.8}$$

$$v^* \;=\; \lim_{n\to\infty}\sup{}^* v_n, \text{ where } \lim_{n\to\infty}\sup{}^* v_n(t,x,i) := \limsup_{\substack{n\,\to\,\infty \\ t'\,\to\,t,\,x'\,\to\,x}} v_n(t',x',i) \tag{3.3.9}$$

(i) We first show the viscosity supersolution property for $v = v_*$. Let (t,x) a point in $[0,T)\times\mathbb{R}^d$, $i\in\mathcal{I}$ and $(p,q,M)\in\bar{J}^-v(t,x,i)$. By (3.3.8) and Lemma 6.1 in [23], there exists sequences

$$n_k \;\to\; \infty, \quad (p_k,q_k,M_k) \;\in\; J^-v_{n_k}(t_k,x_k,i),$$

such that

$$(t_k,x_k,v_{n_k}(t_k,x_k,i),p_k,q_k,M_k) \;\;\to\;\; (t,x,v(t,x,i),p,q,M). \tag{3.3.10}$$

From the viscosity supersolution property for v_{n_j}, we have for all k

$$\begin{aligned}- p_k - b(x_k,i).q_k - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top(x_k,i)M_k) - f(x_k,i,(v_{n_k}(t_k,x_k,j))_j,\sigma^\top(x_k,i)q_k)&\\ -n_k\int_{\mathcal{I}} h^-(i,x_k,v_{n_k}(t_k,x_k,i),v_{n_k}(t_k,x_k,j),\sigma^\top(x_k,i)q_k,j)\lambda(dj) \;\;\geq\;\; 0&\end{aligned}\tag{3.3.11}$$

Let us check that the following inequality holds :

$$\min_{j\in\mathcal{I}} h(i,x_k,v_{n_k}(t_k,x_k,i),v_{n_k}(t_k,x_k,j),\sigma^\top(x_k,i)q_k,j) \;\;\geq\;\; 0. \tag{3.3.12}$$

We argue by contradiction, and assume there exists some $j_0\in\mathcal{I}$ s.t.

$$h(i,x_k,v_{n_k}(t_k,x_k,i),v_{n_k}(t_k,x_k,j),\sigma^\top(x_k,i)q_k,j_0) \;\;<\;\; 0.$$

Then, by continuity of h in all its variables, its nonincreasing property and (3.3.10), one may find some $\varepsilon>0$ such that for all k large enough :

$$h(i,x_k,v_{n_k}(t_k,x_k,i),v_{n_k}(t_k,x_k,j),\sigma^\top(x_k,i)q_k,j) \;\;\leq\;\; -\varepsilon.$$

This implies

$$\int_{\mathcal{I}} h^-(i,x_k,v_{n_k}(t_k,x_k,i),v_{n_k}(t_k,x_k,j),\sigma^\top(x_k,i)q_k,j)\lambda(dj) \;\;\geq\;\; \varepsilon\lambda(j_0) \;>\; 0.$$

By sending k to infinity into (3.3.11), we get the required contradiction. On the other hand, by (3.3.11), we have

$$-p_k - b(x_k,i).q_k - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top(x_k,i)M_k) - f(x_k,i,(v_{n_k}(t_k,x_k,j))_j,\sigma^\top(x_k,i)q_k) \;\;\geq\;\; 0,$$

so that by sending k to infinity :

$$-p - b(x,i).q - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top(x,i)M) - f(x,i,(v(t,x,j))_j,\sigma^\top(x,i)q) \;\;\geq\;\; 0,$$

which proves, together with (3.3.12), that v is a viscosity supersolution to (3.3.2).
(ii) We conclude by showing the viscosity subsolution property for v^*. Let (t,x) a point in $[0,T)\times\mathbb{R}^d$, $i\in\mathcal{I}$ and $(p,q,M)\in\bar{J}^+v^*(t,x,i)$ such that

$$\min_{j\in\mathcal{I}} h(i,x_k,v_{n_k}(t_k,x_k,i),v_{n_k}(t_k,x_k,j),\sigma^\top(x_k,i)q_k,j) \;\;>\;\; 0. \tag{3.3.13}$$

From (3.3.9) and Lemma 6.1 in [23], there exists sequences

$$n_k \ \to \ \infty, \quad (p_k, q_k, M_k) \ \in \ J^+ v_{n_k}(t_k, x_k, i),$$

such that

$$(t_k, x_k, v_{n_k}(t_k, x_k, i), p_k, q_k, M_k) \quad \to \quad (t, x, v^*(t, x, i), p, q, M). \tag{3.3.14}$$

By continuity of the function h, and definition of v^*, we also have

$$\begin{aligned}\limsup_{j\to\infty} h(i, x_k, v_{n_k}(t_k, x_k, i), v_{n_k}(t_k, x_k, j), \sigma^\top(x_k, i) q_k, j) \quad &\leq \\ h(i, x, v(t, x, i), v(t, x, j), \sigma^\top(x, i) q, j), \qquad & \forall\, j \in \mathcal{I}.\end{aligned} \tag{3.3.15}$$

Now, from the viscosity subsolution property for v_{n_k}, we have for all k

$$\begin{aligned}- p_k - b(x_k, i).q_k - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top(x_k, i) M_k) - f(x_k, i, v_{n_k}(t_k, x_k, i), \sigma^\top(x_k, i) q_k) \\ -n_k \int_{\mathcal{I}} h^-(i, x_k, v_{n_k}(t_k, x_k, i), v_{n_k}(t_k, x_k, j), \sigma^\top(x_k, i) q_k, j) \lambda(dj) \quad \leq 0.\end{aligned} \tag{3.3.16}$$

From (3.3.13)-(3.3.14)-(3.3.15), continuity and nonincreasing property of h, we have for k large enough

$$h(i, x_k, v_{n_k}(t_k, x_k, i), v_{n_k}(t_k, x_k, j), \sigma^\top(x_k, i) q_k, j) \quad > \quad 0, \quad \forall j \in \mathcal{I},$$

and so

$$\int_{\mathcal{I}} h^-(i, x_k, v_{n_k}(t_k, x_k, i), v_{n_k}(t_k, x_k, j), \sigma^\top(x_k, i) q_k, j) \lambda(dj) \quad = \quad 0.$$

Hence, by taking the limit as k goes to infinity, into (3.3.16), we conclude that

$$-p - b(x, i).q - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top(x, i) M) - f(x, i, (v^*(t, x, j))_j, \sigma^\top(x, i) q) \quad \leq \quad 0,$$

which shows the viscosity subsolution property for v^* to (3.3.2).

• *Viscosity property on* $\{T\} \times \mathbb{R}^d$. (i) Let first consider the supersolution property of v_* to (3.3.3). Let $x_0 \in \mathbb{R}^d$ and $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d \times \mathcal{I})$ such that (T, x_0) is a global minimum of $(v_* - \varphi)(., i)$ for all $i \in \mathcal{I}$. Passing to the limit the viscosity properties of the penalized BSDE, we naturally derive, as done below

$$\min_{j\in\mathcal{I}} h(i, x, v_*(., i), v_*(., j), \sigma^T D_x\varphi(., i), j)](T, x_0) \quad \geq \quad 0\,,\ i \in \mathcal{I}\,. \tag{3.3.17}$$

Furthermore $v_n(T, .) = g$, $n \in \mathbb{N}$, so that the monotonic property of $(v_n)_n$ leads to $v_*(T, .)$ $\geq g$. Therefore v_* is a viscosity supersolution to (3.3.3).

(ii) We now turn to subsolution property of v^*. Let argue by contradiction and assume the existence of $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{I}$ and $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d \times \mathcal{I})$ such that

$$0 \ = \ (v^* - \varphi)(T, x_0, j) \quad = \quad \max_{[0,T]\times\mathbb{R}^d} (v^* - \varphi)(., j) \quad \forall j \in \mathcal{I}\,, \tag{3.3.18}$$

and we have

$$\min\Big[\varphi(T,x_0,i_0)-g(x_0,i_0)\ ,\ \min_{j\in\mathcal{I}}h(i_0,x,\varphi(.,i_0),\varphi(.,j),D_x\varphi(.,i),j)](T,x_0)\Big] \quad =: \quad 2\varepsilon \ > \ 0.$$

From the regularity of v^*, φ and $D_x\varphi$ as well as the monotonic property of h, we derive the existence of an open neighborhood $\mathcal{O}$ of $(T,x_0)\in[0,T]\times\mathbb{R}^d$, and $\Upsilon, r>0$ such that for all $(t,x,\eta,\eta')\in\mathcal{O}\times(-\Upsilon,\Upsilon)\times B(0,r)$, we have

$$\min\Big[\varphi(t,x,i)-\eta-g(x,i)\ ,\min_{j\in\mathcal{I}}h(i,x,\varphi(.,i)-\eta,\varphi(.,j),\sigma^T[D_x\varphi(.,i)+\eta'],j)](t,x)\Big] \geq \varepsilon, \qquad (3.3.19)$$

Let $(t_k,x_k)_k$ be a sequence in $[0,T)\times\mathbb{R}^d$ satisfying $(t_k,x_k)\to(T,x_0)$ and $v(t_k,x_k,i)\to v^*(T,x_0,i)$. Pick $\delta>0$ such that $[t_k,T]\times B(x_k,\delta)\subset\mathcal{O}$ for k large enough, and introduce the modified test function φ_k given by

$$\varphi_k(t,x,j) \ := \ \varphi(t,x,j)+\left(\zeta\frac{|x-x_k|^2}{\delta^2}+C_k\phi\left(\frac{x-x_k}{\delta}\right)+\sqrt{T-t}\right)\mathbf{1}_{j=i_0},$$

where $0<\zeta<\Upsilon\wedge\delta r$, ϕ is a regular function in $C^2(\mathbb{R}^d)$ such that $\phi|_{\bar{B}(0,1)}\equiv 0$, $\phi|_{\bar{B}(0,1)^c}>0$ and $\lim_{|x|\to\infty}\frac{\phi(x)}{1+|x|}=\infty$, and $C_k>0$ is a constant to be determined precisely later on. We deduce from (3.3.18) that $(v^*-\varphi_k)(t,x,j)\leq-\zeta$, for $(t,x,j)\in[t_k,T]\times\partial B(x_k,\delta)\times\mathcal{I}$. Choosing C_k large enough, the particular form of the function ϕ leads to

$$(v^*-\varphi_k)(t,x,j) \ \leq \ -\frac{\zeta}{2} \qquad \text{for } (t,x,j)\in B(x_k,\delta)^c\times[t_k,T]\times\mathcal{I}. \qquad (3.3.20)$$

Thanks to the $\sqrt{T-t}$ term in the modified test function φ_k, we deduce that

$$-\frac{\partial\varphi_k}{\partial t}(t,x,i_0)-\mathcal{L}^{i_0}\varphi_k(t,x,i_0)-f(i_0,x,(\varphi_k(t,x,j)-\eta\mathbf{1}_{j=i_0})_{j\in\mathcal{I}},\sigma^\top D_x\varphi_k(t,x,i) \geq 0, \qquad (3.3.21)$$

for any $(t,x,\eta)\in[t_k,T)\times B(x_k,\delta)\times(-\Upsilon+\zeta,\Upsilon)$ and k large enough. Choose now $\eta<\Upsilon\wedge\frac{\zeta}{2}\wedge\varepsilon$ and introduce the stopping time

$$\theta_k \ := \ \inf\Big\{s\geq t_k\ ;\ X^k_s\notin B(x_k,\delta)\quad\text{or}\quad I^k_s\neq I^k_{s^-}\Big\}\wedge T\,,$$

where $X^k:=X^{t_k,i,x_k}$ and $I^k:=I^{t_k,i}$. Let finally introduce the process (Y^k,Z^k,U^k,K^k) defined on $[t_k,\theta_k]$ by

$$\left\{\begin{array}{rcl}
Y^k_s & := & \Big[\varphi_k(s,X^k_s,I^k_s)-\eta\Big]\mathbf{1}_{s\in[t_k,\theta_k)}+v(\theta_k,X^k_{\theta_k},I^k_s)\mathbf{1}_{s=\theta_k}\,,\\
Z^k_s & := & \sigma^\top(X^k_s,I^k_{s^-})D_x\varphi_k(s,X^k_s,I^k_{s^-})\,,\\
U^k_s & := & \Big(\Big[\varphi(s,X^k_s,j)-[\varphi_k(s,X^k_s,I^k_{s^-})-\eta]\Big]\mathbf{1}_{j\neq i_0}\Big)_{j\in\mathcal{I}}\,,\\
K^k_s & := & -\int_{t_k}^s\Big\{\Big(\frac{\partial\varphi_k}{\partial t}+\mathcal{L}^{i_0}\varphi_k\Big)(r,X^k_r,I^k_r)+f(I^k_r,X^k_r,(\varphi_k(r,X^k_r,j)-\eta\mathbf{1}_{j=i_0})_{j\in\mathcal{I}},Z^k_s)\Big\}dr\\
 & & -\int_{t_k}^s\int_{\mathcal{I}}(\varphi_k-\eta-\varphi)(r,X^k_r,j)\mu(dr,dj)+\ [\varphi_k-\eta-v]\,(\theta_k,X^k_{\theta_k},I^k_{\theta_k})\mathbf{1}_{s=\theta_k}.
\end{array}\right.$$

One easily checks from (3.3.19)-(3.3.20)-(3.3.21) that (Y^k,Z^k,U^k,K^k) is a solution to (3.3.6)-(3.3.7) on $[t_k,\theta_k]$. By Lemma 3.3.2, we deduce that

$$\varphi_k(t_k,x_k,i)-\eta \ = \ \varphi(t_k,x_k,i)+\sqrt{T-t_k}-\eta \ \geq \ v(t_k,x_k,i)\,, \qquad \text{for all } k \text{ large enough}\ .$$

Letting k go to infinity, this contradicts (3.3.18) and concludes the proof. □

The main drawback of this result identifying the value function associated to the minimal solution Y to (3.2.2)-(3.2.3) as a solution to the system of variational inequalities (3.3.2)-(3.3.3) is that it requires the strong assumption **(H1)**. Following from similar arguments as Proposition 6.3 in [46], the next remark provides a nice sufficient condition for **(H1)**.

Remark 3.3.2 Assume that there exists a Lipschitz function $w \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{I})$ satisfying a linear growth condition, supersolution to (3.3.3) and such that

$$\mathcal{L}^i w(.,i) + f(\cdot,(w(.,j))_{j\in\mathcal{I}}, \sigma^\top Dw(.,i)) \quad \leq \quad C, \quad \text{ on } \mathbb{R}^d,$$

for some constant C. Then **(H1)** holds true.

3.4 Uniqueness result

The purpose of this section is to characterize the value function v as the unique viscosity solution to the system of variational inequalities (3.3.2)-(3.3.3). The proof relies as usual on the obtention of a comparison theorem presented below and generalizes the Theorem 4.3 in [46]. In particular, this implies the continuity of v and, as a consequence, the strong convergence results by penalization and the minimality condition presented respectively in sections 3.2.3 and 3.2.4 of the paper.

3.4.1 Assumptions

We shall require the following additional assumptions.

(H2) There exists a nonnegative function $\Lambda \in \mathcal{C}^2(\mathbb{R}^d \times \mathcal{I})$ and a positive constant ρ satisfying

(i) $\mathcal{L}^i\Lambda(.,i) + f(i,.,(\Lambda(.,j))_{j\in\mathcal{I}}, \sigma^\top D_x\Lambda(.,i)) \leq \rho\Lambda(.,i)$, for all $i \in \mathcal{I}$,

(ii) $\min_{j\in\mathcal{I}} h(i,x,\Lambda(x,i),\Lambda(x,j),\sigma^\top D_x\Lambda(x,i),j) > 0$, for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^d$,

(iii) $\Lambda(x,i) \geq g(x,i)$, for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^d$,

(iv) $\lim_{|x|\to\infty} \frac{\Lambda(x,i)}{1+|x|} = \infty$, for all $i \in \mathcal{I}$.

As in Bouchard [10], this assumption allows to to construct a nice strict supersolution to (3.3.2) leading to a control on solutions to (3.3.2)-(3.3.3) by convexe perturbations. Nevertheless, as in [46], the dependence of the driver f and the constraint h with respect to some components (Y,Z,U) of the solution to the BSDE, forces us to add some extra convexity conditions.

(H3)

(i) The function $f(i, x, ., .)$ is convex in $((y_i)_{i\in\mathcal{I}}, z) \in \mathbb{R}^{\mathcal{I}} \times \mathbb{R}^d$ for all $(x, i) \in \mathbb{R}^d \times \mathcal{I}$.

(ii) The function $h(i, x, ., j)$ is concave in $(y_i, y_j, z) \in [\mathbb{R}]^2 \times \mathbb{R}^d$ a for all $(x, i, j) \in \mathbb{R}^d \times [\mathcal{I}]^2$.

(iii) The function $f(i, x, y, z, u_1, \ldots, u_{i-1}, ., u_{i+1}, \ldots, u_m)$ is increasing for all $(i, x, y, z, (u_j)_{j\in\mathcal{I}-\{i\}}) \in \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}]^{m-1}$

(iv) The function $h(i, x, ., z, u, j)$ is decreasing for all $(i, j, x, z, u) \in [\mathcal{I}]^2 \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$

Remark 3.4.1 In the classical case of optimal switching case where f is independent of y and z and h is of the form $(i, u, j) \mapsto -u - c(i, j)$, Assumption **(H3)** is automatically satisfied. Furthermore, as in [10], **(H2)** holds under an additional assumption on the structure of the constraint, specifying the existence of a sequence $(d_i)_{i\in\mathcal{I}}$ such that $c(i, j) < d_i - d_j$ for all $(i, j) \in \mathcal{I} \times \mathcal{I}$ such that $i \neq j$. Indeed, one easily checks that, in this case, the function $\Lambda : (x, i) \mapsto \alpha + |x|^2 + d_i$ with α large enough satisfies **(H2)**.

3.4.2 The comparison theorem

Theorem 3.4.1 *Assume that* **(H0)**, **(H1)**, **(H2)** *and* **(H3)** *hold. Then, for any U (resp. V) lsc (resp. usc) viscosity supersolution (resp. subsolution) to* (3.3.2)-(3.3.3) *satisfying a linear growth condition :*

$$\sup_{(t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathcal{I}} \frac{|U(t, x, i)| + |V(t, x, i)|}{1 + |x|} \quad < \quad \infty\,,$$

we have $U \geq V$ on $[0, T] \times \mathbb{R}^d \times \mathcal{I}$. In particular, the function v in (3.2.7) *is the unique viscosity solution to* (3.3.2)-(3.3.3) *satisfying a linear growth condition, and $v(., i)$ is continuous on $[0, T) \times \mathbb{R}^d$, for all $i \in \mathcal{I}$.*

Proof. • *Comparison principle.* As usual, we shall argue by contradiction by assuming that

$$\sup_{[0,T]\times\mathbb{R}^d\times\mathcal{I}} (V - U) > 0\,. \tag{3.4.1}$$

1. For some $\eta > 0$ to be chosen below, let

$$\hat{U}(t, x, i) := e^{(\rho+\eta)t}U(t, x, i)\ ,\ \hat{V}(t, x, i) := e^{(\rho+\eta)t}V(t, x, i) \ \text{ and } \ \hat{\Lambda}(t, x, i) := e^{(\rho+\eta)t}\Lambda(x, i).$$

A straightforward derivation shows that $\hat{U}$ (resp. $\hat{V}$) is a viscosity supersolution (resp. subsolution) to

$$\begin{aligned}\min\Big[\rho w - \frac{\partial w}{\partial t} - \mathcal{L}^i w - \hat{f}\left(., w, \sigma^\top D_x w\right),&\\ \min_{j\in\mathcal{I}} \hat{h}\left(., w, w(., j), \sigma^\top D_x w, j\right)\Big](t, x, i) \quad &= \quad 0\,,\end{aligned} \tag{3.4.2}$$

$$\min\Big[w - \hat{g}, \min_{j\in\mathcal{I}} \hat{h}(., w, w(., j), \sigma^\top D_x w, j)\Big](T^-, x, i) \quad = \quad 0\,, \tag{3.4.3}$$

respectively on $[0,T)\times\mathbb{R}^d\times\mathcal{I}$ and $\mathbb{R}^d\times\mathcal{I}$, where

$$\begin{aligned}\hat{f}(t,i,x,r_1,\ldots,r_m,q) &:= e^{(\rho+\eta)t}\tilde{f}\left(i,x,r_1e^{-(\rho+\eta)t},\ldots,r_me^{-(\rho+\eta)t},qe^{-(\rho+\eta)t}\right)-\eta r\,,\\ \hat{h}(i,t,x,r_i,r_j,q,j) &:= e^{(\rho+\eta)t}\tilde{h}(i,x,e^{-(\rho+\eta)t},r_ie^{-(\rho+\eta)t},r_je^{-(\rho+\eta)t},qe^{-(\rho+\eta)t},j)\end{aligned}$$

and $\hat{g}(x,i)=e^{(\rho+\eta)T}g(x,i)$, for all $(t,x,r,q,i,j)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{I}\times\mathcal{I}$. Since $\tilde{f}$ is Lipschitz, we can choose η large enough so that $\hat{f}(i,.)$ is nonincreasing in r_i. Denote $\hat{W}:=(1-\alpha)\hat{U}+\alpha\hat{\Lambda}$ with $\alpha\in(0,1)$. By (3.4.1) and the growth condition **(H2)**(iv) of Λ, we have

$$\sup_{[0,T]\times\mathbb{R}^d\times\mathcal{I}}(\hat{V}-\hat{W})=(\hat{V}-\hat{W})(t_0,x_0,i_0)>0. \tag{3.4.4}$$

for some $(t_0,x_0,i_0)\in[0,T]\times\mathbb{R}^d\times\mathcal{I}$ and α small enough. Moreover from the viscosity supersolution property (3.4.2)-(3.4.3) of $\hat{U}$, and the conditions **(H2)**(i), (ii), **(H3)**(i), (ii), (iii), we see that $\hat{W}$ is a viscosity supersolution to

$$\begin{aligned}\rho w(.,i)-\frac{\partial w}{\partial t}(.,i)-\mathcal{L}^iw(.,i)-\hat{f}\left(i,.,w,\sigma^\top D_xw(.,i)\right) &\geq 0, \text{ on } [0,T)\times\mathbb{R}^d,\\ \min_{j\in\mathcal{I}}\hat{h}\left(i,.,w(.,i),w(.,j),\sigma^\top D_xw(.,i),j\right) &\geq \alpha\hat{q}(.,i), \text{ on } [0,T]\times\mathbb{R}^d,\end{aligned}$$

for all $i\in\mathcal{I}$, where $\hat{q}(t,x,i):=e^{(\rho+\eta)t}\min_{j\in\mathcal{I}}\tilde{h}\left(i,x,\Lambda(x,i),\Lambda(x,j),\sigma^\top D_x\Lambda(x,i),j\right)$ is positive on $[0,T]\times\mathbb{R}^d\times\mathcal{I}$ by **(H2)**(ii).

2. Denote for all $(t,x,y,i)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{I}$ and $n\geq 1$

$$\Theta_n(t,x,y,i) := \hat{V}(t,x,i)-\hat{W}(t,y,i)-\varphi_n(t,x,y,i),$$

with

$$\varphi_n(t,x,y,i) := n|x-y|^2+|x-x_0|^4+|t-t_0|^2+|i-i_0|.$$

By the growth assumption on U and V and **(H2)**(iv), for all n, there exists $(t_n,x_n,y_n,i_n)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{I}$ attaining the maximum of Θ_n on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{I}$. By standard arguments, we have :

$$(t_n,x_n,y_n,i_n) \to (t_0,x_0,x_0,i_0), \tag{3.4.5}$$

$$n|x_n-y_n|^2 \to 0, \tag{3.4.6}$$

$$\hat{V}(t_n,x_n,i_n)-\hat{W}(t_n,y_n,i_n) \to \hat{V}(t_0,x_0,i_0)-\hat{W}(t_0,x_0,i_0). \tag{3.4.7}$$

3. Using the uppersemicontinuity of $\hat{V}$, the compactness of $\mathcal{I}$, and properties of the functions h and Λ, we obtain by the same arguments as in [46]

$$\min_{j\in\mathcal{I}}\hat{h}(i_n,.,\hat{V}(.,i_n),\hat{V}(.,j),\sigma^\top D_x\varphi_n(.,y_n,i_n),j))(t_n,x_n) > 0\,, \tag{3.4.8}$$

for n large enough.

4. Let us check that, up to a subsequence, $t_n < T$ for all $n \in \mathbb{N}$. Assume on the contrary that $t_n = t_0 = T$ for n large enough, and deduce from (3.4.8) and the viscosity subsolution property of $\hat{V}$ to (3.4.3) that

$$\hat{V}(T, x_n, i_n) \quad \leq \quad \hat{g}(x_n, i_n).$$

On the other hand, by the viscosity supersolution property of $\hat{U}$ to (3.4.3) and **(H2)**(iii), we have $\hat{W}(T, y_n, i_n) \geq \hat{g}(y_n, i_n)$, which leads to

$$\hat{V}(T, x_n, i_n) - \hat{W}(T, y_n, i_n) \quad \leq \quad \hat{g}(x_n, i_n) - \hat{g}(y_n, i_n).$$

Sending n to infinity, the continuity of $\hat{g}$ implies $(\hat{V} - \hat{W})(t_0, x_0, i_0) \leq 0$ which contradicts (3.4.4).

5. We may then apply Ishii's lemma (see Theorem 8.3 in [23]) to $(t_n, x_n, y_n) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ that attains the maximum of $\Theta_n(., i_n)$, for all $n \geq 1$: there exist $(p^n_{\tilde{V}}, q^n_{\tilde{V}}, M_n) \in \bar{J}^{2,+}\tilde{V}(t_n, x_n, i_n)$ and $(p^n_{\tilde{W}}, q^n_{\tilde{W}}, N_n) \in \bar{J}^{2,-}\tilde{W}(t_n, y_n, i_n)$ such that

$$\begin{aligned} p^n_{\tilde{V}} - p^n_{\tilde{W}} \quad &= \quad \partial_t \varphi_n(t_n, x_n, y_n, i_n) \quad = \quad 2(t_n - t_0), \\ q^n_{\tilde{V}} \quad &= \quad D_x \varphi_n(t_n, x_n, y_n, i_n), \qquad q^n_{\tilde{W}} \quad = \quad -D_y \varphi_n(t_n, x_n, y_n, i_n), \end{aligned}$$

and

$$\begin{pmatrix} M_n & 0 \\ 0 & -N_n \end{pmatrix} \leq A_n + \frac{1}{2n} A_n^2, \tag{3.4.9}$$

where $A_n = D^2_{(x,y)} \varphi_n(t_n, x_n, y_n, i_n)$. From the viscosity supersolution property of $\tilde{W}$ to (3.4.5), we have

$$\begin{aligned} \rho \tilde{W}(t_n, y_n, i_n) - p^n_{\tilde{W}} - b(y_n, i_n) \cdot D_y \varphi(t_n, x_n, y_n, i_n) - \frac{1}{2} \mathrm{tr}(\sigma\sigma^\top(y_n, i_n) N_n) \\ - \tilde{f}(t_n, y_n, i_n, (\tilde{W}(t_n, y_n, j))_j, -\sigma^\top(y_n, i_n) D_y \varphi(t_n, x_n, y_n, i_n)) \quad \geq \quad 0. \end{aligned}$$

On the other hand, from (3.4.8) and the viscosity subsolution property of $\tilde{V}$ to (3.4.2), we have

$$\begin{aligned} \rho \tilde{V}(t_n, x_n, i_n) - p^n_{\tilde{V}} + b(x_n, i_n) \cdot D_x \varphi(t_n, x_n, y_n, i_n) - \frac{1}{2} \mathrm{tr}(\sigma\sigma^\top(x_n, i_n) M_n) \\ - \tilde{f}(t_n, x_n, i_n, (\tilde{V}(t_n, x_n, j))_j, \sigma^\top(x_n, i_n) D_x \varphi(t_n, x_n, y_n, i_n)) \quad \leq \quad 0. \end{aligned}$$

By substracting the two previous inequalities, we obtain

$$\begin{aligned} \rho(\tilde{V}(t_n, x_n, i_n) - \tilde{W}(t_n, y_n, i_n)) \quad &\leq \\ p^n_{\tilde{V}} - p^n_{\tilde{W}} + \Delta F_n& \\ - \Big(b(x_n, i_n) \cdot D_x \varphi_n(t_n, x_n, y_n, i_n) + b(y_n, i_n) \cdot D_y \varphi_n(t_n, x_n, y_n, i_n) \Big)& \\ + \frac{1}{2} \mathrm{tr} \Big(\sigma\sigma^\top(x_n, i_n) M_n - \sigma\sigma^\top(y_n, i_n) N_n \Big)& \quad , \end{aligned} \tag{3.4.10}$$

where

$$\begin{aligned}\Delta F_n &= \tilde{f}(t_n, x_n, i_n, (\tilde{V}(t_n, x_n, j))_j, \sigma^\top(x_n, i_n) D_x \varphi_n(t_n, x_n, y_n, i_n)) \\ &\quad - \tilde{f}(t_n, y_n, i_n, (\tilde{W}(t_n, y_n, j))_j, -\sigma^\top(y_n, i_n) D_y \varphi_n(t_n, x_n, y_n, i_n)).\end{aligned}$$

From (3.4.5), we have $p_{\tilde{V}}^n - p_{\tilde{W}}^n \to 0$ as n goes to infinity. From the Lipschitz property of b, and (3.4.6), we have

$$\lim_{n\to\infty} \Big(b(x_n, i_n) \cdot D_x \varphi_n(t_n, x_n, y_n, i_n) + b(y_n, i_n) \cdot D_y \varphi_n(t_n, x_n, y_n, i_n) \Big) = 0.$$

As usual, from (3.4.9), (3.4.5), (3.4.6), and the Lipschitz property of σ, we have

$$\limsup_{n\to\infty} \ \mathrm{tr}\Big(\sigma\sigma^\top(x_n, i_n) M_n - \sigma\sigma^\top(y_n, i_n) N_n \Big) \leq 0.$$

Moreover, using the nonincreasing property of $\tilde{f}$ in its third variable, and the Lipschitz property of $\tilde{f}$, we have from (3.4.5)-(3.4.6)-(3.4.7)

$$\limsup_{n\to\infty} \Delta F_n \leq 0.$$

Therefore, by sending $n \to \infty$ into (3.4.10), we conclude with (3.4.7) that $\rho(\tilde{V} - \tilde{W})(t_0, x_0, i_0) \leq 0$, a contradiction with (3.4.4).

• *Uniqueness for v.* The uniqueness result is a direct consequence of the comparison principle, and the continuity of $v(., i)$ on $[0, T) \times \mathbb{R}^d$ for all $i \in \mathcal{I}$ follows from the fact that in this case $v_*(., i) = v^*(., i)$. □

Remark 3.4.2 Without comparison theorem for (3.3.2)-(3.3.3), one can still characterize v with the PDE (3.3.2)-(3.3.3) in the case where the IPDE (3.3.1) admits a comparison principle. Following the arguments of the proof of Proposition 3.3 in [65], one can prove that v is indeed the minimal viscosity solution to (3.3.2)-(3.3.3) in the class of functions with linear growth.

Remark 3.4.3 Considering the particular case of a BSDE of the form (3.2.2)-(3.2.3) without jump component, the previous theorem offers a nice uniqueness property for PDEs representing solutions of constrained BSDEs derived in [65].

3.5 Numerical issues

This section is dedicated to the numerical implications of the Feynman Kac representation of coupled systems of variational inequalities. In order to approach the solution to the corresponding constrained BSDE with jump, we focus here on the approximation of the corresponding penalized BSDE with jumps. We combine the discrete time scheme introduced by [12] with the statistical estimation projection presented in [38]. This gives rise to a convergent probabilistic algorithm solving coupled systems of variational inequalities. In all the section, we suppose that Assumptions **(H0)**, **(H1)**, **(H2)** and **(H3)** are satisfied.

Classical numerical resolution of quasi variational inequalities relies on the use of iterated free boundary or optimal stopping problems presented in [60]. We first solve the semi-linear PDE without free boundary condition. We then consider the same PDE combined with the boundary condition caracterized by the previous value function. Iterating this procedure, the algorithm converges to the solution of quasi variational inequalities. In a switching problem, the algorithm corresponds to constraining the solution associated to $n+1$ possible switches by the obstacle built with the solution where only n switches are allowed. Such a numerical approach is computationally demanding, since it requires, at each induction step n, the resolution of an optimal stopping problem. Moreover, at step $n+1$, in order to determine the solution at one point, one needs to compute the function at time n in the whole space due to the nonlocal form of the obstacle.

We first add an assumption on g classicaly used for discretisation of BSDEs.

(H4) The functions g, f and h are Lipschitz continuous in x : there exists a constant k such that

$$\begin{aligned} |g(x,i) - g(x',i)| + |f(i,x,y,z,(u_j)_j) - f(i,x',y,z,(u_j)_j)| & \\ +|h(i,x,y,z,u_i,u_j,j) - h(i,x',y,z,u_i,u_j,j)| & \leq k|x-x'|, \end{aligned}$$

for all $(i,j,x,x',y,z,(u_j)_j) \in \mathcal{I}^2 \times [\mathbb{R}^d]^2 \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}$.

We propose here a numerical approach based on the probabilistic representation of the solution to the QVI (3.3.2) by the constrained BSDE given by

$$\begin{aligned} Y_t &= g(I_T,X_T) + \int_t^T f(I_s,X_s^I,Y_s+U_s,Z_s)ds + K_T - K_t \\ & \quad - \int_t^T Z_s.dW_s - \int_t^T \int_{\{1,\dots,m\}} U_s(i)\mu(ds,di), \end{aligned} \tag{3.5.1}$$

together with the constraint

$$h(I_{t-},X_t^I,Y_{t-},Y_{t-}+U_t(j),Z_t,j) \quad \geq \quad 0, \quad j \in \{1,\dots,m\}\,. \tag{3.5.2}$$

The algorithm divides in three steps.

Step 1. Approximation by penalization. We first approach the constrained BSDE (3.5.1)-(3.5.2) by penalization as in section 3 of the paper. Given $n \in \mathbb{N}$, the problem is now to etimate the solution of

$$Y_t^n = g(X_T,I_T) + \int_t^T f_n(I_s,X_s,Y_s^n+U_s^n,Z_s^n)ds - \int_t^T Z_s^n dW_s - \int_t^T\int_{\mathcal{I}} U_s^n(j)\tilde{\mu}(dj,ds), \tag{3.5.3}$$

where, for any $(i,x,y,z,u) \in \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathbb{I}}$, we have

$$f_n(i,x,y,z,u) := f(i,x,y+u,z) - \int_{\mathcal{I}} [u(j) + nh^-(i,x,y,y+u(j),z,j)]\lambda(dj).$$

According to Proposition 3.2.1 and Remark 3.2.2, we have the following control on the penalization error:

$$\|Y - Y^n\|_{\mathcal{S}^2} + \|Z - Z^n\|_{\mathbf{L^2_W}} + \|U - U^n\|_{\mathbf{L^2_{\tilde{\mu}}}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.5.4}$$

Step 2. Time discretization. Observe that the pure Jump process I can be simulated perfectly and denote by (τ_j) its jump times on $[0,T]$. Let introduce the Euler time scheme approximation X^h of the forward process X defined on the concatenation $(s_l)_k$ of the regular time grid $\{t_i := ih,\ i = 1,\dots,T/h\}$ with the jumps (τ_j) of J:

$$X_0^h = X_0 \quad \text{and} \quad X^h_{s_{k+1}} := X^h_{s_k} + b(I_{s_k}, X^h_{s_k})(s_{k+1} - s_k) + \sigma(I_{s_k}, X^h_{s_k})[W_{s_{k+1}} - W_{s_k}].$$

We deduce an approximation $Y_T^{n,h}$ of Y_T^n at maturity given by $g(X_T^h, I_T)$. The penalized BSDE (3.2.4) can now be discretized by an extension of the scheme exposed by Bouchard and Elie [12]. An approximation of Y^n at time 0 is computed recursively following the backward scheme for $i = T/h - 1, \cdots, 0$:

$$\begin{cases} Z_{t_i}^{n,h} & := & \frac{1}{h}\mathbb{E}_{t_i}\left[Y_{t_{i+1}}^{n,h}(W_{t_{i+1}} - W_{t_i})\right]; \\ U_{t_i}^{n,h}(j) & := & \frac{1}{h}\mathbb{E}_{t_i}\left[Y_{t_{i+1}}^{n,h}\frac{\tilde{\mu}((t_i,t_{i+1}]\times\{j\})}{\lambda(j)}\right], \quad j = 1,\cdots,m; \\ Y_{t_i}^{n,h} & := & \mathbb{E}_{t_i}\left[Y_{t_{i+1}}^{n,h} + \int_{t_i}^{t_{i+1}} f_n(I_s, X_{t_i}^h, Y_{t_{i+1}}^{n,h}, Z_{t_i}^{n,h}, U_{t_i}^{n,h})ds\right], \end{cases} \tag{3.5.5}$$

where $\mathbb{E}_{t_i}$ denotes the conditional expectation with respect to $\mathcal{G}_{t_i}$. Following the arguments in section 2.5 of [12] and identifying $(Y^{n,h}, Z^{n,h}, U^{n,h})$ as a constant by part process on each interval $(t_i, t_{i+1}]$, one can verify the convergence of this time-discretization approximation :

$$\|Y^n - Y^{n,h}\|_{\mathcal{S}^2} + \|Z^n - Z^{n,h}\|_{\mathbf{L^2_W}} + \|U^n - U^{n,h}\|_{\mathbf{L^2_{\tilde{\mu}}}} \underset{h\to\infty}{\longrightarrow} 0, \quad n \in \mathbb{N}. \tag{3.5.6}$$

Step 3. Approximation of the conditional expectations. The last step consists in estimating the conditional expectation operators $\mathbb{E}_{t_i}$ arising in (3.5.5). There are several methods proposed in the literature relying on quantization method, basis function approximation, cubature formulas or Malliavin Calculus. We adopt here the approach of Longstaff-Schwarz [52] generalized by [38] and [31] relying on least square regressions and Monte Carlo simulations.

Fix $N \in \mathbb{N}$ and simulate N independent copies of the Brownian increments $(W^k_{t_{i+1}} - W^k_{t_i})_{0\le i\le T/h}$ and the poisson measure $(\tilde{\mu}^k((t_i,t_{i+1}]\times\mathcal{I})_{0\le i\le T/h}$. For each simulation $k \le N$, define I_k^N and $X_k^{h,N}$ the trajectories of I and X^h. By induction, one can easily verify the Markov property of the process $(Y^{n,h}, Z^{n,h}, U^{n,h})$ defined in (3.5.5):

$$Y_{t_i}^{n,h} = \xi_i^{n,h}(I_{t_i}, X_{t_i}^h), \quad Z_{t_i}^{n,h} = \phi_i^{n,h}(I_{t_i}, X_{t_i}^h), \quad U_{t_i}^{n,h} = \psi_i^{n,h}(I_{t_i}, X_{t_i}^h),$$

for some deterministic functions $\xi_i^{n,h}, \phi_i^{n,h}, \psi_i^{n,h}$. The idea is to approximate these functions using Ordinary Least Square (OLS) estimators, as detailed in [31]. Given $L \in \mathbb{N}$, we introduce a collection of basis functions $(\xi_l^L, \phi_l^L, \psi_l^L)_{1\leq l\leq L}$ of $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$. For each trajectory $k \leq N$, define the associated terminal value given by $Y_{k,t_N}^{n,h,L,N} := g(I_{k,t_N}^N, X_{k,t_N}^{h,N})$. Now we define recursively $(Z_{k,t_i}^{n,h,L,N}, U_{k,t_i}^{n,h,L,N})$, backward in time for $i = T/h-1, \cdots, 0$, by computing OLS approximations as follows:

$$(\hat{\alpha}_1, \cdots, \hat{\alpha}_L) := \arg\min_{\alpha_1,\cdots,\alpha_L} \frac{1}{N}\sum_{k=1}^N \Big|\frac{1}{h}Y_{k,t_{i+1}}^{n,h,L,N}[W_{t_{i+1}}^k - W_{t_i}^k] - \sum_{l=1}^L \alpha_l \phi_l^L(I_{k,t_i}^N, X_{k,t_i}^{h,N})\Big|^2,$$

$$(\hat{\beta}, \cdots, \hat{\beta}_L)(j) := \arg\min_{\beta_1,\cdots,\beta_L} \frac{1}{N}\sum_{k=1}^N \Big|\frac{1}{h}Y_{t_{i+1}}^{n,h,L,N}\frac{\tilde{\mu}^k((t_i,t_{i+1}]\times\{j\})}{\lambda(j)} - \sum_{l=1}^L \beta_l \psi_l^L(I_{k,t_i}^N, X_{k,t_i}^{h,N})\Big|^2,$$

leading to the approximation

$$Z_{k,t_i}^{n,h,L,N} := \sum_{l=1}^L \hat{\alpha}_l \phi_l^L(I_{k,t_i}^N, X_{k,t_i}^{h,N}) \quad \text{and} \quad U_{k,t_i}^{n,h,L,N}(j) := \sum_{l=1}^L \hat{\beta}_l(j)\psi_l^L(I_{k,t_i}^N, X_{k,t_i}^{h,N}), \quad j \in \mathcal{I}.$$

It remains to introduce $(\hat{\gamma}_1, \cdots, \hat{\gamma}_L)$ the minimizer of the mean square error

$$\frac{1}{N}\sum_{k=1}^N \Big|Y_{k,t_{i+1}}^{n,h,L,N} + \int_{t_i}^{t_{i+1}} f_n(I_{k,s}^N, X_{k,t_i}^{h,N}, Y_{t_{i+1}}^{n,h,L,N}, Z_{t_i}^{n,h,L,N}, U_{t_i}^{n,h,L,N})ds - \sum_{l=1}^L \gamma_l \xi_l^L(I_{k,t_i}^N, X_{k,t_i}^{h,N})\Big|^2;$$

in order to deduce the OLS approximation $Y_{k,t_i}^{n,h,X^{h,NL,N}} := \sum_{l=1}^L \hat{\gamma}_l \xi_l^L(I_{k,t_i}^N, X_{k,t_i}^{h,N})$. We refer to [38] and [31] for the control of the statistical error due to the approximation of the conditional expectation operators by OLS projections. Adapting their arguments to our context, we can verify that the statistical error

$$\|Y^{n,h} - Y^{n,hL,N}\|_{\mathcal{S}^2} + \|Z^{n,h} - Z^{n,h,L,N}\|_{\mathbf{L^2_W}} + \|U^{n,h} - U^{n,h,L,N}\|_{\mathbf{L^2_{\tilde{\mu}}}} \underset{N,L\to\infty}{\longrightarrow} 0 \tag{3.5.7}$$

for $n \in \mathbb{N}$ and $h > 0$.

The global error of the algorithm is controlled by (3.5.4), (3.5.6) and (3.5.7) and converges to 0. In order to get a rate of convergence for the algorithm, we need to study the dependence in n of all the estimates coming from the discretization and the statistical error. Furthermore, one requires to control the penalization error which seems to be very intricate. This very challenging point is left for further research.

Chapter 4

A discrete-time approximation for multidimensional BSDEs with oblique reflections

Abstract : In this paper, we study the discrete-time approximation of multi-dimensional reflected BSDEs presented by Hu and Tang [44]. In comparison to the penalizing approach followed by Hamadène and Jeanblanc [39] or Elie and Kharroubi [32], we study a more natural scheme based on oblique projections. We provide a control on the error of the algorithm by introducing and studying the notion of multidimensional discretely reflected BSDE. In the particular case where the driver does not depend on the variable Z, the error on the grid points is of order $\frac{1-\varepsilon}{2}$ for all $\varepsilon > 0$, and $\frac{1}{2}$ in the case of constant cost functions.

Keywords: BSDE with oblique reflections, discrete time approximation, Switching problems.

4.1 Introduction

Trying to optimize the productivity of a power station with start-up and switch-down costs, Hamadène and Jeanblanc [39] study the Snell envelope representation for optimal switching problems in continuous time. Observing that the difference between the two value functions starting in both modes of production is solution of a doubly reflected BSDE, they derive existence and uniqueness of solution to this problem. Nevertheless their approach restricts to optimal switching problems with only two possible modes of production. The extension to optimal switching problem in high dimension is studied by Carmona and Ludkovski [18], Porchet, Touzi and Warin [70] and by Pham, Ly Vath and Zhou [68] for an infinite time horizon consideration. In these papers, the resolution of optimal switching problem relies mostly on its link with systems of variational inequalities. The rigorous derivation of a multidimensional BSDE representation for this type of problem is obtained by Djehiche, Hamadène and Popier [28] and Hu and Tang [44]. In this latter paper, they introduce the more general notion of multi-dimensional reflected BSDE, with dynamics of the form

$$\begin{cases} \dot{Y}^i_t = g^i(X_T) + \int_t^T f^i(X_s, \dot{Y}^i_s, \dot{Z}^i_s)\mathrm{d}s - \int_t^T \dot{Z}^i_s \mathrm{d}W_s + \dot{K}^i_T - \dot{K}^i_t, \\ \dot{Y}^i_t \geq \max_{j\in\mathcal{I}}\{\dot{Y}^j_t - c_{i,j}\}, \quad 0 \leq t \leq T\ , \\ \int_0^T [\dot{Y}^i_t - \max_{j\in\mathcal{I}}\{\dot{Y}^j_t - c_{i,j}\}]d\dot{K}^i_t = 0\ , \quad i \in \mathcal{I}\ , \end{cases} \tag{4.1.1}$$

where $\mathcal{I} := \{1, \ldots, d\}$, f and g are Lipschitz functions and X is the solution of a forward stochastic differential equation. Hu and Tang [44] derive existence and uniqueness of solution to BSDE (4.1.1) and relate it to optimal switching problems where the driver f depends on the $\dot{Y}$ and $\dot{Z}$ components of the BSDE solution. All the components of the $\dot{Y}$ process are interconnected so that the vector $\dot{Y}$ lies in a closed convex set $\mathcal{C}$ characterized by the costs $c_{i,j}$. The vector process $\dot{Y}$ is reflected obliquely on the boundaries of the domain. The main contribution of this paper is to offer and study a numerical scheme for the discrete-time approximation of the solution to multidimensional reflected BSDE of the form (4.1.1).

The key for the discrete time approximation of solution to BSDEs relies in the path regularity of the process Z derived by Zhang in [80]. Based on this beautiful estimate, Bouchard and Touzi [14] and Gobet, Lemor and Warin [37] introduced respectively an implicit and explicit discrete time scheme for the resolution of classical BSDEs. Being given a time discretization grid of $[0, T]$ with time mesh $|\pi|$ and an Euler approximation X^π of X, the scheme computes backward in time some approximation of the solution to the BSDE, with an optimal quadratic error of order $|\pi|^{1/2}$. As detailed in the recent survey [13], several extensions of this type of scheme allow in particular for the addition of jumps [12] or normal reflections [11, 21, 22]. Since the approximation procedure always relies on a backward induction, we shall regroup these schemes into the generic name of moonwalk scheme. We intend here to extend this so-called moonwalk scheme to the consideration of oblique reflections appearing in (4.1.1). Recently two of the authors related in [32] the solution of (4.1.1) to corresponding constrained BSDEs with jumps. As presented in [33], this type of BSDE can be numerically approximated combining a penalization procedure with the use of the moonwalk scheme for BSDEs with jumps. Unfortunately, no convergence rate is available for this algorithm. We present here a more natural scheme based on oblique reflections on the boundary of the convex set $\mathcal{C}$.

For this purpose, we introduce a grid $\Re = \{r_0, \ldots, r_\kappa\}$ of possible reflection points, simply extracted from the discretization grid. We then consider, as in [11, 21] but adapted to our context, the following discretely obliquely reflected BSDE : $Y_T = \widetilde{Y}_T := g(X_T) \in \mathcal{C}$, and for $j \leq \kappa - 1$ and $t \in [r_j, r_{j+1})$

$$\left\{ \begin{array}{rcl} \widetilde{Y}_t & = & Y_{r_{j+1}} + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}_u, Z_u)\mathrm{d}u - \int_t^{r_{j+1}} Z_u \mathrm{d}W_u \ , \\ Y_t & = & \widetilde{Y}_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(\widetilde{Y}_t)\mathbf{1}_{\{t \in \Re\}}. \end{array} \right. \tag{4.1.2}$$

where $\mathcal{P}$ is the oblique projection operator on $\mathcal{C}$. We then consider its natural discrete-time scheme defined by

$$Y_T^\pi \ := \ g(X_T^\pi) \ ,$$

and for $i \in \{0 \ldots, n-1\}$

$$\left\{ \begin{array}{rcl} \bar{Z}_{t_i}^\pi & := & (t_{i+1} - t_i)^{-1} \mathbb{E}\left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i})' \mid \mathcal{F}_{t_i} \right] \ , \\ \widetilde{Y}_{t_i}^\pi & := & \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \ , \\ Y_{t_i}^\pi & := & \widetilde{Y}_{t_i}^\pi \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(\widetilde{Y}_{t_i}^\pi)\mathbf{1}_{\{t_i \in \Re\}} \ , \end{array} \right. \tag{4.1.3}$$

Using classical argument, we obtain the convergence of this discrete time scheme to (4.1.2). However, the obliqueness of the projections on $\mathcal{C}$ rules out classical methods (used e.g. in [21] for the case of normal reflections) to get a bound on a convergence rate.

To overcome this difficulty, we interpret, as Hu and Tang [44], the solution of the BSDE with discrete oblique reflection (4.1.1) as the value process of an optimal switching problem with switching times belonging to $\Re$. Introducing a convenient process dominating both (4.1.3) and (4.1.2) and using a comparison argument, we are able to retrieve a bound of the convergence rate of $|\pi|^{1/2}$ for the value process, in the case where f does not depend on the variable Z. Using the same approach, we prove, when f is bounded in the variable Z, the convergence of the discretely reflected BSDE (4.1.2) to the continuously reflected BSDE (4.1.1) at a rate $|\Re|^{\frac{1-\varepsilon}{2}}$ for all $\varepsilon > 0$. Combining those two results, we then obtain the convergence of (4.1.3) to (4.1.1) for the value process at a rate of $|\pi|^{\frac{1-\varepsilon}{2}}$ on the grid for all $\varepsilon > 0$. In the case where the cost functions are constant this convergence holds at a rate $|\pi|^{\frac{1}{2}}$ on the grid, and $|\pi|^{\frac{1}{4}}$ on [0,T], see Theorem 4.4.3.

All these results are obtained without any assumption on the non-degeneracy of the volatility matrix σ.

The rest of the paper is organized as follows. In Section 2, we introduce the notion of discretely reflected BSDEs, connect it with optimal switching problems. Section 3 focuses on the discrete time approximation of discretely reflected BSDEs. We present the discrete time scheme and study its convergence. In the particular case where the driver function does not involve the variable Z, we provide a rate of convergence. Finally, Section 4 study the extension to the continuously reflected case. In the case where the driver f is bounded in the variable Z, we provide a rate of convergence of the discretely reflected BSDE to the continuously one.

Notations. Throughout this paper we are given a finite time horizon T and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t \geq 0}$. Any element $x \in \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$ will be identified to a column vector with i-th component x^i and Euclidian norm $|x|$. For $x, y \in \mathbb{R}^\ell$, $x.y$ denotes the scalar product of x and y, and x' denotes the transpose of x. We denote by $\succeq$ the component by component partial ordering relation on vectors. $\mathcal{M}^{m,d}$ denotes the set of real matrices with m lines and d columns. We denotes by C_b^k the set of functions from $\mathbb{R}^d$ to $\mathbb{R}$ with continuous and bounded derivatives up to order k. $\mathcal{S}^{\mathbf{p}}$ (resp. $\mathcal{S}^{\mathbf{c},\mathbf{p}}$), $p \geq 1$, is the set of real-valued *càdlàg*[1] (resp. continuous) $\mathbb{F}$-adapted processes $Y = (Y_t)_{0 \leq t \leq T}$ such that

$$\|Y\|_{\mathcal{S}^{\mathbf{p}}} \quad := \quad \mathbb{E}\Big[\sup_{0 \leq t \leq T} |Y_t|^p\Big]^{\frac{1}{p}} \quad < \quad \infty.$$

$\mathcal{H}^{\mathbf{p}}$, $p \geq 1$, is the set of $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable processes $Z = (Z_t)_{0 \leq t \leq T}$ such that

$$\|Z\|_{\mathcal{H}^{\mathbf{p}}} \quad := \quad \mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \quad < \quad \infty.$$

$\mathbf{A}^{\mathbf{p}}$ (resp. $\mathbf{A}^{\mathbf{c},\mathbf{p}}$), $p \geq 1$, is the closed subset of $\mathcal{S}^{\mathbf{p}}$ (resp. $\mathcal{S}^{\mathbf{c},\mathbf{p}}$) consisting of nondecreasing processes $K = (K_t)_{0 \leq t \leq T}$ with $K_0 = 0$. In the following, we shall use these notations without specifying the dimension or the dependence in $\omega \in \Omega$ when it is clearly given by the context.

4.2 Discretely obliquely reflected BSDE

In this section, we define and study discretely obliquely reflected BSDEs. In particular, we show how its solution relates to the solution of a one-dimensional optimal switching problem, where the switching times are restricted to a discrete set of time.

4.2.1 Definition

Let $T > 0$ be a given time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis supporting a d-dimensional Brownian motion. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by the Brownian motion is supposed to satisfy the usual conditions.

[1] The french acronym for *continu à droite limité à gauche* meaning right continuous with left limits.

Let X be the solution on $[0,T]$ to the following Stochastic Differential Equation

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s \ , \quad 0 \le t \le T \ , \tag{4.2.1}$$

where $X_0 \in \mathbb{R}^m$ and $b : \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \to \mathcal{M}^{m,d}(\mathbb{R})$ are L-Lipschitz functions, for some positive constant L.

In the sequel we denote by C_L a constant whose value may change from line to line but which depends only on L. We use the notation C_L^p whenever it depends on some other parameter $p > 0$.

Under the above assumption, the following estimates are well known (see e.g. [50])

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] \le C_L^p \text{ and } \sup_{s\in[0,T]} \left(\mathbb{E}\left[\sup_{u\in[0,T],|u-s|\le h} |X_s - X_u|^p\right]\right)^{\frac{1}{p}} \le C_L^p \sqrt{h} \ . \tag{4.2.2}$$

We then introduce a family $(\mathcal{C}(x))_{x\in\mathbb{R}^m}$ of closed convex domains:

$$\mathcal{C}(x) := \left\{ y \in \mathbb{R}^d \mid y^i \ge \max_j (y^j - c_{i,j}(x)) \, , \quad \forall i \in \mathcal{I} \right\} \ , \tag{4.2.3}$$

recalling that $\mathcal{I} = \{1, \ldots, d\}$. The maps $(c_{i,j})_{1\le i,j\le d}$, $c_{i,j} : \mathbb{R}^m \to \mathbb{R}^+$, are L-Lipschitz continuous and satistfy:

$$\begin{cases} c_{i,i}(.) = 0 \, , & \text{for } 1 \le i \le d \, ; \\ \inf_{x\in\mathbb{R}^m} c_{i,j}(x) > 0 \, , & \text{for } 1 \le i,j \le d \, , \text{ with } i \ne j \, ; \\ \inf_{x\in\mathbb{R}^m} \{c_{i,j}(x) + c_{j,l}(x) - c_{i,l}(x)\} > 0, & \text{for } 1 \le i,j,l \le d, \text{ with } i \ne j, \, j \ne l. \end{cases} \tag{4.2.4}$$

Remark 4.2.1 As detailled in [41], these conditions allow to get existence and uniqueness of a solution to the corresponding continuously reflected BSDE, see Section 4 hereafter for more details.

We consider $\mathcal{P}$ an oblique projection operator on $\mathcal{C}$ w.r.t. y defined by

$$\mathcal{P} : (x,y) \in \mathbb{R}^m \times \mathbb{R}^d \ \mapsto \ \left(\max_j \left\{ y^j - c_{i,j}(x) \right\} \right)_{1\le i\le d} .$$

We present in the next Lemma some useful properties of $\mathcal{P}$.

Lemma 4.2.1 *The operator $\mathcal{P}$ is Lipschitz. For all $x \in \mathbb{R}^m$, $\mathcal{P}(x,.)$ is an oblique projection onto $\mathcal{C}(x)$ and it is increasing with respect to the partial ordering relation $\succeq$, where $y \succeq y'$ means $y^i \ge (y')^i$ for all $i \in \mathcal{I}$.*

Proof. Observe first that obviously $\mathcal{P}(x,y)=y$ for $y\in\mathcal{C}(x)$ and $x\in\mathbb{R}^m$. It follows from the structure condition (4.2.4) on the maps $(c_{ij})_{1\le i,j\le d}$ that

$$\mathcal{P}(x,y)^i \;=\; \max_j\left\{y^j-c_{i,j}(x)\right\} \;\ge\; \max_j\left\{y^j-c_{k,j}(x)\right\}-c_{i,k}(x) \;=\; \mathcal{P}(x,y)^k-c_{i,k}(x)\,,$$

for $(x,y)\in\mathbb{R}^m\times\mathbb{R}^d$ and $1\le i,k\le d$. Therefore $\mathcal{P}(x,.)$ is an oblique projection on $\mathcal{C}(x)$. We compute that

$$\begin{aligned}
|\mathcal{P}(x_1,y_1)-\mathcal{P}(x_2,y_2)| &= \left(\sum_{i=1}^d\left|\max_j\,(y_1^j-c_{i,j}(x_1))-\max_j\,(y_2^j-c_{i,j}(x_2))\right|^2\right)^{1/2}\\
&\le \sqrt{d}\max_j\,|y_1^j-y_2^j|+|c_{i,j}(x_1)-c_{i,j}(x_2)|\\
&\le L\Big(|y_1-y_2|+|x_1-x_2|\Big)\,, \qquad\qquad y_1,y_2\in\mathbb{R}^d\,,\; x_1,x_2\in\mathbb{R}^m\,.
\end{aligned}$$

$\square$

In the spirit of [11, 22], we introduce the notion of multidimensional discretely obliquely reflected BSDE characterized by

- a partition grid $\Re:=\{r_0=0,\dots,r_\kappa=T\}$ of $[0,T]$ satisfying

$$|\Re|:=\max_{1\le k\le\kappa}|r_k-r_{k-1}|\le\frac{L}{\kappa}, \tag{4.2.5}$$

- the previous convex set valued function $\mathcal{C}$,
- an L-Lipschitz function $g:\mathbb{R}^m\to\mathbb{R}^d$ such that $g(x)\in\mathcal{C}(x)$ for all $x\in\mathbb{R}^m$,
- a generator function, i.e. an L-lipschitz map $f:\mathbb{R}^m\times\mathbb{R}^d\times\mathcal{M}^{d,d}(\mathbb{R})\to\mathbb{R}^d$. We also assume that the component i of $f(t,y,z)$ depends only on the component i of the vector y and on the line i of the matrix z i.e. $f^i(t,y,z)=f^i(t,y^i,z^i)$. This allows to interpret Y as the value process of an optimal switching of BSDEs (see [44]).

For a given data $(\Re,\mathcal{C},g,f)$, a discretely obliquely reflected BSDE is a triplet $(\widetilde{Y},Y,Z)\in(\mathcal{S}^{\mathbf{2}}\times\mathcal{S}^{\mathbf{2}}\times\mathcal{H}^{\mathbf{2}})^{\mathcal{I}}$ satisfying $Y_T=\widetilde{Y}_T:=g(X_T)\in\mathcal{C}(X_T)$, and, defined in a backward manner, for $j\le\kappa-1$ and $t\in[r_j,r_{j+1})$, by

$$\left\{\begin{array}{rcl}\widetilde{Y}_t &=& Y_{r_{j+1}}+\int_t^{r_{j+1}}f(X_u,\widetilde{Y}_u,Z_u)\mathrm{d}u-\int_t^{r_{j+1}}Z_u.\mathrm{d}W_u\ ,\\ Y_t &=& \widetilde{Y}_t\mathbf{1}_{\{t\notin\Re\}}+\mathcal{P}(X_t,\widetilde{Y}_t)\mathbf{1}_{\{t\in\Re\}}.\end{array}\right. \tag{4.2.6}$$

This rewrites equivalently on $[0,T]$ as

$$\widetilde{Y}_t=g(X_T)+\int_t^T f(X_u,\widetilde{Y}_u,Z_u)\mathrm{d}u-\int_t^T Z_u.\mathrm{d}W_u+(\widetilde{K}_T-\widetilde{K}_t)\,,\quad 0\le t\le T\,, \tag{4.2.7}$$

where the nondecreasing (for the partial ordering $\succeq$) process $\widetilde{K} \in (\mathbf{A^2})^{\mathcal{I}}$ is defined by

$$\widetilde{K}_t := \sum_{r \in \Re \setminus \{0\}} \Delta \widetilde{K}_r \, \mathbf{1}_{\{r \leq t\}} \text{ and } \Delta \widetilde{K}_t = Y_t - \widetilde{Y}_t = -(\widetilde{Y}_t - \widetilde{Y}_{t-}), \qquad 0 \leq t \leq T. \quad (4.2.8)$$

Observe that Y and $\widetilde{Y}$ differ only on the grid points of $\Re$. On each interval of the form $[r_k, r_{k+1})$, $(\widetilde{Y}, Z)$ is solution to a classical non reflected BSDE with terminal condition $Y_{r_{k+1}}$, see [61]. Therefore, existence and uniqueness of the discretely reflected BSDE follows directly from a concatenation of the solutions on all the grid intervals.

4.2.2 Corresponding optimal switching problem

In this section, we interpret the solution of the discretely obliquely RBSDE (4.2.7) as the value process of a corresponding optimal switching problem, where the possible times of switch are restricted to belong the grid $\Re$. Our approach relies on similar arguments as the one followed by Hu and Tang [44] in a framework with continuous reflections.

A switching strategy a is a nondecreasing sequence of stopping times $(\theta_j)_{j \in \mathbb{N}}$, combined with a sequence of random variables $(\alpha_j)_{j \in \mathbb{N}}$ valued in $\mathcal{I}$, such that α_j is $\mathcal{F}_{\theta_j}$−measurable, for any $j \in \mathbb{N}$. We denote by $\mathcal{A}$ the set of such strategies. For $a = (\theta_j, \alpha_j)_{j \in \mathbb{N}} \in \mathcal{A}$, we introduce N^a the (random) number of switch before T:

$$N^a \quad = \quad \#\{k \in \mathbb{N}^* \ : \ \theta_k \leq T\} \, .$$

To any switching strategy $a = (\theta_j, \alpha_j)_{j \in \mathbb{N}} \in \mathcal{A}$, we associate the current state process $(a_t)_{t \in [0,T]}$ and the compound cost process $(A^a_t)_{t \in [0,T]}$ defined respectively by

$$a_t \ := \ \alpha_0 \mathbf{1}_{\{0 \leq t \leq \theta_0\}} + \sum_{j=1}^{N^a} \alpha_{j-1} \mathbf{1}_{\{\theta_{j-1} < t \leq \theta_j\}} \qquad \text{and} \qquad A^a_t \ := \ \sum_{j=1}^{N^a} c_{\alpha_{j-1}, \alpha_j}(X_{\theta_j}) \mathbf{1}_{\{\theta_j \leq t \leq T\}} \, ,$$

for $0 \leq t \leq T$. For $(t,i) \in [0,T] \times \mathcal{I}$, $\mathcal{A}_{t,i}$ the set of admissible strategies starting from i at time t is defined by

$$\mathcal{A}_{t,i} = \{a = (\theta_j, \alpha_j)_j \in \mathcal{A} \ | \theta_0 = t, \ \alpha_0 = i, \ \mathbb{E}\big[|A^a_T|^2\big] < \infty\}$$

We denote by $\mathcal{A}^{\Re}_{t,i}$ the set of $\Re$−admissible strategies:

$$\mathcal{A}^{\Re}_{t,i} \quad := \quad \{ \ a = (\theta_j, \alpha_j)_{j \in \mathbb{N}} \in \mathcal{A}_{t,i} \quad | \quad \theta_j \in \Re \ , \ \forall j \leq N^a \ \} \ .$$

For $(t,i) \in [0,T] \times \mathcal{I}$, and $a \in \mathcal{A}^{\Re}_{t,i}$, we consider as in [44] the associated one dimensional switched BSDE defined by

$$U^a_u = g^{a_T}(X_T) + \int_u^T f^{a_s}(X_s, U^a_s, V^a_s) \mathrm{d}s - \int_t^T V^a_s . \mathrm{d}W_s - A^a_T + A^a_u \, , \quad t \leq u \leq T \, . \ (4.2.9)$$

Theorem 3.1 in [44] interprets the components of the solution to the continuously reflected BSDE (4.1.1) as the Snell envelop associated to switched processes of the form (4.2.9), where the switching strategies a are not restricted to lie in the reflection grid $\Re$. The next theorem is a version of this Snell envelop representation to the discretely reflected BSDE (4.2.7).

Theorem 4.2.1 *For any $i \in \mathcal{I}$ and $t \in [0,T]$, the following holds:*

(i) The process $\widetilde{Y}$ dominates any switched BSDE, i.e.

$$U_t^a \quad \leq \quad \widetilde{Y}_t^i \qquad \mathbb{P}-a.s.\,, \qquad \textit{for any } a \in \mathcal{A}_{i,t}^{\Re}\,. \tag{4.2.10}$$

(ii) Define the strategy $a^* = (\theta_j^*, \alpha_j^*)_{j\geq 0}$ recursively by $(\theta_0^*, \alpha_0^*) := (t,i)$ and, for $j \geq 1$,

$$\begin{aligned} \theta_j^* \quad &:= \quad \inf\left\{ s \in [\theta_{j-1}^*, T] \cap \Re \;\middle|\; \widetilde{Y}_s^{\alpha_{j-1}^*} \leq \max_{k \neq \alpha_{j-1}^*} \left\{ \widetilde{Y}_s^k + c_{\alpha_{j-1}^*,k}(X_s) \right\} \right\}, \\ \alpha_j^* \quad &:= \quad \min\left\{ k \neq \alpha_{j-1}^* \;\middle|\; \widetilde{Y}_{\theta_j^*}^k + c_{\alpha_{j-1}^*,k}(X_{\theta_j^*}) = \max_{\ell \neq \alpha_{j-1}^*} \left\{ \widetilde{Y}_s^\ell + c_{\alpha_{j-1}^*,\ell}(X_{\theta_k^*}) \right\} \right\}. \end{aligned}$$

Then we have $a^* \in \mathcal{A}_{t,i}^{\Re}$ and

$$\widetilde{Y}_t^i = U_t^{a^*} \;\; \mathbb{P}-a.s.\,. \tag{4.2.11}$$

(iii) The following "Snell envelop" representation holds

$$\widetilde{Y}_t^i \quad = \quad essup_{a \in \mathcal{A}_{t,i}^{\Re}} U_t^a\,, \qquad \mathbb{P}-a.s.\,. \tag{4.2.12}$$

Proof. Assertion (iii) is a direct consequence of (i) and (ii).

Step 1. We first prove (i). Let fix $t \in [0,T]$ and $i \in \mathcal{I}$. Set $a = (\theta_k, \alpha_k)_{k\geq 0} \in \mathcal{A}_{t,i}^{\Re}$ and the process $(\widetilde{Y}^a, Z^a)$ defined on $[t,T]$ by

$$\begin{aligned} \widetilde{Y}_s^a \;&:=\; \sum_{k\geq 0} \widetilde{Y}_s^{\alpha_k} \mathbf{1}_{\{\theta_k \leq s < \theta_{k+1}\}} + g^{a_T}(X_T)\mathbf{1}_{\{s=T\}} \;\text{ and} \\ Z_s^a \;&:=\; \sum_{k\geq 0} Z_s^{\alpha_k} \mathbf{1}_{\{\theta_k \leq s < \theta_{k+1}\}}\,, \quad t \leq s \leq T\,. \end{aligned} \tag{4.2.13}$$

These processes jump between the components of the discretely reflected BSDE (4.2.6) and, between two jumps, we have

$$\begin{aligned} \widetilde{Y}_{\theta_k}^{\alpha_k} \;&=\; Y_{\theta_{k+1}}^{\alpha_k} + \int_{\theta_k}^{\theta_{k+1}} f^{\alpha_k}(X_s, \widetilde{Y}_s^{\alpha_k}, Z_s^{\alpha_k})\mathrm{d}s - \int_{\theta_k}^{\theta_{k+1}} Z_s^{\alpha_k}.\mathrm{d}W_s + \widetilde{K}_{\theta_{k+1}-}^{\alpha_k} - \widetilde{K}_{\theta_k}^{\alpha_k} \\ &=\; \widetilde{Y}_{\theta_{k+1}}^a + \int_{\theta_k}^{\theta_{k+1}} f^{a_s}(X_s, \widetilde{Y}_s^a, Z_s^a)\mathrm{d}s - \int_{\theta_k}^{\theta_{k+1}} Z_s^a.\mathrm{d}W_s + \widetilde{K}_{\theta_{k+1}-}^{\alpha_k} - \widetilde{K}_{\theta_k}^{\alpha_k} \\ &\quad + (Y_{\theta_{k+1}}^{\alpha_k} - \widetilde{Y}_{\theta_{k+1}}^{\alpha_{k+1}})\,, \qquad\qquad k \geq 0\,. \end{aligned} \tag{4.2.14}$$

Introducing

$$\widetilde{K}^a_s := \sum_{k=0}^{N^a-1} [\int_{(\theta_k\wedge s,\theta_{k+1}\wedge s)} d\widetilde{K}^{\alpha_k}_s + \mathbf{1}_{\{\theta_{k+1}\le s\}}(Y^{\alpha_k}_{\theta_{k+1}} - \widetilde{Y}^{\alpha_{k+1}}_{\theta_{k+1}} + c_{\alpha_k,\alpha_{k+1}}(X_{\theta_{k+1}}))] \,,$$

for $s \in [t,T]$, and summing up (4.2.14) over k, we get

$$\begin{aligned} \widetilde{Y}^a_u &= g^{a_T}(X_T) + \int_u^T f^{a_s}(X_s,\widetilde{Y}^a_s,Z^a_s)\mathrm{d}s - \int_u^T Z^a_s.\mathrm{d}W_s \\ &\quad -A^a_T + A^a_u + \widetilde{K}^a_T - \widetilde{K}^a_u \,, \quad t \le u \le T \,. \end{aligned} \tag{4.2.15}$$

Using the equality $Y_{\theta_k} = \mathcal{P}(X_{\theta_k}, \widetilde{Y}_{\theta_k})$ for all $k \in \{0,\dots,N^a\}$, we obtain that $\widetilde{K}^a$ is increasing. Since U^a solves (4.2.9), we deduce by a comparison argument (see Theorem 1.3 in [64]) that $U^a_t \le \widetilde{Y}^a_t$, proving (4.2.10).

Step 2. We now consider the strategy a^*, recall (ii), and the associated processes $\widetilde{Y}^{a^*}$ and Z^{a^*} defined as in (4.2.13). By definition of a^*, we have

$$Y^{\alpha^*_k}_{\theta^*_{k+1}} = \left[\mathcal{P}\left(X_{\theta^*_{k+1}}, \widetilde{Y}_{\theta^*_{k+1}}\right)\right]^{\alpha^*_k} = \widetilde{Y}^{\alpha^*_{k+1}}_{\theta^*_{k+1}} - c_{\alpha^*_k,\alpha^*_{k+1}}(X_{\theta^*_{k+1}}) \,, \quad k \ge 0 \,,$$

which gives

$$\int_{(\theta^*_k,\theta^*_{k+1})} d\widetilde{K}^{\alpha^*_k}_s = 0 \ \text{ and } \ Y^{\alpha^*_k}_{\theta^*_{k+1}} - \widetilde{Y}^{\alpha^*_k}_{\theta^*_{k+1}} + c_{\alpha^*_k,\alpha^*_{k+1}}(X_{\theta^*_{k+1}}) = 0 \,, \tag{4.2.16}$$

for all $k \in \{0,\dots,N^{a^*}-1\}$. We deduce from (4.2.15) that

$$\widetilde{Y}^{a^*}_u = g^{a^*_T}(X_T) + \int_u^T f^{a^*_s}(X_s,\widetilde{Y}^{a^*}_s,Z^{a^*}_s)ds - \int_u^T Z^{a^*}_s.\mathrm{d}W_s - A^{a^*}_T + A^{a^*}_u \,, \quad t \le u \le T \,.$$

Hence $(\widetilde{Y}^{a^*}, Z^{a^*})$ and (U^{a^*}, V^{a^*}) are solutions of the same BSDE and $\widetilde{Y}^i_t = U^{a^*}_t$. We finaly observe that $a^* \in \mathcal{A}^{\Re}$, i.e. $\mathbb{E}|A^{a^*}_T|^2 < \infty$. Indeed, we have

$$A^{a^*}_T = \widetilde{Y}^{a^*}_T - \widetilde{Y}^{a^*}_t + \int_t^T f^{a^*_s}(X_s,\widetilde{Y}^{a^*}_s,Z^{a^*}_s)\mathrm{d}s - \int_t^T Z^{a^*}_s.\mathrm{d}W_s + A^{a^*}_t \,. \tag{4.2.17}$$

By definition of a^* and the structure condition on the cost (4.2.4), we have $|A^{a^*}_t| \le \max_{k\neq i}|c_{i,k}(X_t)|$ which gives $\mathbb{E}[|A^{a^*}_t|^2] \le C_L$. Then, using the fact that $(\widetilde{Y}, Z)$ belongs to $\mathcal{S}^\mathbf{2} \times \mathcal{H}^\mathbf{2}$, and the properties of the generator f, we get from (4.2.17) the square integrability of $A^{a^*}_T$. □

Remark 4.2.2 Although the optimal strategy a^* depends on the initial parameters t and i, we omit the script (t,i) for ease of notation.

Remark 4.2.3 Notice that Theorem 4.2.1 holds true for a random generator $(f(t,.,.))_t$ such that $\mathbb{E}\Big[\int_0^T |f(s,0,0)|^2 ds\Big] < \infty$, and random costs $(c_{i,j}(t))_t$, satisfying (4.2.4) and $c_{i,j} \in \mathcal{S}^\mathbf{p}$, for $i,j \in \mathcal{I}$.

4.2.3 Some a priori estimates

We present here some estimates on the processes $\widetilde{Y}$, Z and their associated switched processes U^{a^*},V^{a^*} and A^{a^*}. The proof of the following Proposition is postponed to the appendix.

Proposition 4.2.1 *The following holds*

$$\sum_{i=1}^{d}\|\widetilde{Y}^i\|_{\mathcal{S}^{\mathbf{p}}}+\|Z^i\|_{\mathcal{H}^{\mathbf{p}}}+\|\widetilde{K}^i\|_{\mathcal{S}^{\mathbf{p}}}\leq C_L^p\ , \quad p\geq 2\ ,$$

recalling that C_L^p does not depend on $\Re$.

Using the link between (U^{a^*},V^{a^*}) and $(\widetilde{Y},Z)$, we obtain the same result for U^{a^*}, V^{a^*} and A^{a^*}.

Corollary 4.2.1 *The following bound holds*

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|U_s^{a^*}|^p\Big]+\mathbb{E}\Big[(\int_t^T|V_s^{a^*}|^2ds)^{\frac{p}{2}}\Big]+\mathbb{E}\Big[\sup_{s\in[t,T]}|A_s^{a^*}|^p\Big]+\mathbb{E}[|N^{a^*}|^p]\leq C_L^p\ , \quad p\geq 2\ ,$$

for all $(t,i)\in[0,T]\times\mathcal{I}$, recalling that C_L^p does not depend on $\Re$.

Proof. Fix $p\geq 2$. According to the identification of (U^{a^*},V^{a^*}) with $(\widetilde{Y}^{a^*},Z^{a^*})$, obtained in the proof of Theorem 4.2.1, we get from the previous proposition

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|U_s^{a^*}|^p\Big]+\mathbb{E}\Big[(\int_t^T|V_s^{a^*}|^2ds)^{\frac{p}{2}}\Big]\leq C_L^p\ ,$$

Then, writing the equation satisfied by (U^{a^*},V^{a^*}) and using standard arguments for BSDEs, we get

$$\mathbb{E}\Big[|A_T^{a^*}|^p\Big]\leq C_L(\mathbb{E}\Big[\sup_{s\in[t,T]}|U_s^{a^*}|^p\Big]+\mathbb{E}\Big[(\int_t^T|V_s^{a^*}|^2ds)^{\frac{p}{2}}\Big]+\mathbb{E}[|A_t^{a^*}|^p]$$

By definition of a^* and (4.2.4), we have $|A_t^{a^*}|\leq C_L(1+|X_t|)$ which gives with the previous inequality

$$\mathbb{E}\Big[|A_T^{a^*}|^p\Big]\leq C_L$$

From (4.2.4) we get $C_L\mathbb{E}\big[|N^{a^*}|^p\big]\leq\mathbb{E}\big[|A_T^{a^*}|^p\big]$ which completes the proof. □

4.3 Discrete-time Approximation

We present here a discrete time scheme for the approximation of the solution of the discretely reflected BSDE (4.2.6).

4.3.1 Discrete-time approximation of the forward process

We consider a grid $\pi := \{t_0 = 0, \dots, t_n = T\}$ on the time interval $[0, T]$, with modulus $|\pi|$ ($|\pi| := \max_{0 \le i \le n-1} |t_{i+1} - t_i|$), such that $\Re \subset \pi$ and $|\pi| n \le L$. In the sequel, the process X is approximated by its Euler scheme X^π, whose dynamics are given by

$$X^\pi_t \quad = \quad X_0 + \int_0^t b(X^\pi_{\pi(s)}) \mathrm{d}s + \int_0^t \sigma(X^\pi_{\pi(s)}).\mathrm{d}W_s \; , \quad 0 \le t \le T \; , \tag{4.3.1}$$

where $\pi(t) := \sup\{t_i \in \pi \; ; \; t_i \le t\}$ is defined on $[0, T]$ as the projection to the closest previous grid point of π. We then have the following bound uniform in the grid π:

$$\mathbb{E}\left[\sup_{0 \le t \le T} |X^\pi_t|^p \right]^{1/p} \quad \le \quad C^p_L \;\; , \; p \ge 2 \; . \tag{4.3.2}$$

The control of the error between X and its Euler scheme X^π is well understood, see e.g. [48], and we have

$$\mathbb{E}\left[\sup_{0 \le t \le T} |X_t - X^\pi_t|^p \right]^{1/p} \quad \le \quad C^p_L \, |\pi|^{\frac{1}{2}} \;\; , \; p \ge 2 \; . \tag{4.3.3}$$

4.3.2 An approximation scheme for discretely reflected BSDEs

We introduce an Euler-type approximation scheme for the discretely reflected BSDEs. Starting from the terminal condition

$$Y^\pi_T = \widetilde{Y}^\pi_T := g(X^\pi_T) \in \mathcal{C}(X^\pi_T)$$

we compute recursively, for $i \le n - 1$,

$$\left\{ \begin{array}{lcl} \bar{Z}^\pi_{t_i} & = & (t_{i+1} - t_i)^{-1} \mathbb{E}\left[Y^\pi_{t_{i+1}} (W_{t_{i+1}} - W_{t_i})' \mid \mathcal{F}_{t_i} \right] \; , \\ \widetilde{Y}^\pi_{t_i} & = & \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X^\pi_{t_i}, \widetilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \; , \\ Y^\pi_{t_i} & = & \widetilde{Y}^\pi_{t_i} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X^\pi_{t_i}, \widetilde{Y}^\pi_{t_i}) \mathbf{1}_{\{t_i \in \Re\}}. \end{array} \right. \tag{4.3.4}$$

This kind of backward scheme has been already considered when no reflection occurs, see e.g. [14], and in the reflected case, see e.g. [11, 56, 22]. See also [13] for a recent survey on the subject.

Combining an induction argument with the Lispchitz-continuity of f, g and the projection operator, one easily checks that the above processes are square integrable and that the conditional expectations are well defined at each step of the algorithm.

Remark 4.3.1 This so-called moonwalk algorithm is given by an implicit formulation, and one should use a fixed point argument to compute explicitly $\widetilde{Y}^\pi$ at each grid point. As observed in [37], one could also use an explicit version of the scheme, replacing the second equation in (4.3.4) by

$$\widetilde{Y}^\pi_{t_i} \quad = \quad \mathbb{E}\left[Y^\pi_{t_{i+1}} + (t_{i+1} - t_i) f(X^\pi_{t_i}, \widetilde{Y}^\pi_{t_{i+1}}, \bar{Z}^\pi_{t_i}) \mid \mathcal{F}_{t_i} \right] \; , \qquad i < n \; .$$

For later use, we introduce the piecewise continuous time scheme associated to $(Y^\pi, \widetilde{Y}^\pi, \bar{Z}^\pi)$. By the martingale representation theorem, there exists $Z^\pi \in \mathcal{H}^{\mathbf{2}}$ such that

$$Y^\pi_{t_{i+1}} = \mathbb{E}_{t_i}\Big[Y^\pi_{t_{i+1}}\Big] + \int_{t_i}^{t_{i+1}} Z^\pi_u.\mathrm{d}W_u \ , \quad i \le n-1 \ ,$$

and by the Itô isometry, for $i \le n-1$,

$$\bar{Z}^\pi_{t_i} = \frac{1}{t_{i+1}-t_i}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z^\pi_s \mathrm{d}s \mid \mathcal{F}_{t_i}\right] . \tag{4.3.5}$$

We set

$$\bar{Z}^\pi_t \quad := \quad \bar{Z}^\pi_{t_i} \ \text{ for } \ t \in [t_i, t_{i+1}) \ .$$

We then define $\widetilde{Y}^\pi$ on $[t_i, t_{i+1})$ by

$$\widetilde{Y}^\pi_t = Y^\pi_{t_{i+1}} + (t_{i+1}-t) f(X^\pi_{t_i}, \widetilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) - \int_t^{t_{i+1}} Z^\pi_u.\mathrm{d}W_u \ , \tag{4.3.6}$$

and by

$$Y^\pi_t := \widetilde{Y}^\pi_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X^\pi_t, \widetilde{Y}^\pi_t)\mathbf{1}_{\{t \in \Re\}} \ .$$

This can be rewritten:

$$\left\{ \begin{array}{rcl} \widetilde{Y}^\pi_t & = & g(X^\pi_T) + \int_t^T f(X^\pi_{\pi(u)}, \widetilde{Y}^\pi_{\pi(u)}, \bar{Z}^\pi_u)\mathrm{d}u - \int_t^T Z^\pi_u.\mathrm{d}W_u + (\widetilde{K}^\pi_T - \widetilde{K}^\pi_t) \ , \\ \widetilde{K}^\pi_t & = & \sum_{r \in \Re \setminus \{0\}} \Delta \widetilde{K}^\pi_r \mathbf{1}_{\{r \le t\}} \ \text{ and } \ \Delta \widetilde{K}^\pi_t = Y^\pi_t - \widetilde{Y}^\pi_t = -(\widetilde{Y}^\pi_t - \widetilde{Y}^\pi_{t-}) \ , \\ Y^\pi_t & = & \widetilde{Y}^\pi_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X^\pi_t, \widetilde{Y}^\pi_t)\mathbf{1}_{\{t \in \Re\}} \ , \qquad 0 \le t \le T. \end{array} \right. \tag{4.3.7}$$

4.3.3 Convergence of the discrete-time scheme

The following proposition provides the convergence of the discrete time scheme to the solution of the discretely reflected BSDE (4.2.6).

Proposition 4.3.1 *The discrete-time scheme $(\widetilde{Y}^\pi, \bar{Z}^\pi)$ converges to $(\widetilde{Y}, Z)$:*

$$\sup_{t \in [0,T]} \mathbb{E}\Big[|\widetilde{Y}_t - \widetilde{Y}^\pi_t|^2 + |Y_t - Y^\pi_t|^2\Big] + \mathbb{E}\Big[\int_0^T |Z_s - \bar{Z}^\pi_s|^2 \mathrm{d}s\Big] \quad \longrightarrow \quad 0 \qquad \textit{as} \ \ |\pi| \to 0 \ .$$

Proof. Using the same arguments as in the proof of Proposition 3.4.1 in [21], we get the following inequality:

$$\begin{aligned} \sup_{t \in [0,T]} \mathbb{E}\Big[|\widetilde{Y}_t - \widetilde{Y}^\pi_t|^2 + |Y_t - Y^\pi_t|^2\Big] + \mathbb{E}\Big[\int_0^T |Z_s - \bar{Z}^\pi_s|^2 \mathrm{d}s\Big] \quad \le \\ C_L L^\kappa \left(|\pi| + \mathbb{E}\Big[\int_0^T (|\widetilde{Y}_t - \widetilde{Y}_{\pi(t)}|^2 + |Z_t - \bar{Z}_{\pi(t)}|^2)\mathrm{d}t\Big]\right) \end{aligned} \tag{4.3.8}$$

where $\kappa+1$ is the length of $\Re$: $\Re = \{r_0 = 0, \dots, r_\kappa = T\}$, and

$$\bar{Z}_{t_i} := \frac{1}{t_{i+1}-t_i}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s \mid \mathcal{F}_{t_i}\right], \quad i \in \{0, \dots, n-1\}. \tag{4.3.9}$$

The only difference is that $\mathcal{P}$ here is not 1-Lipschitz but only L-Lipschitz (See step 1.b in the proof of Proposition 3.4.1 in [21]). This justifies the term L^κ in (4.3.8).

Obviously $\mathbb{E}\Big[\int_0^T |\widetilde{Y}_t - \widetilde{Y}_{\pi(t)}|^2 \mathrm{d}t\Big] \longrightarrow 0$ as $|\pi| \downarrow 0$ and since, $\bar{Z}$ is the best approximation in $\mathcal{H}^2$ of Z by adapted processes constant on $[t_i, t_{i+1})$, it follows from (4.3.8) that the scheme is convergent. □

In the general case, we are not able to retrieve an explicit control on the convergence rate of the scheme in term of $|\pi|$. This is usually done by studying the process Z and its regularity property. This problem is left for further research. In the next section, we provide a suitable control in term of $|\pi|$ but under restrictions on the function f.

4.3.4 The case where f does not depend on Z

In this section, we improve substantially the results of Proposition 4.3.1 when the driver function f does not depend on the variable Z.

Theorem 4.3.1 *If f does not depend on Z, the following holds:*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|\widetilde{Y}_t - \widetilde{Y}^\pi_t\right|^2\right] \le C_L|\pi| ,$$

for all π such that $|\pi|L < 1$.

In order to compare the processes $\widetilde{Y}$ and $\widetilde{Y}^\pi$, recall (4.2.7) and (4.3.7), we introduce the process $(\hat{Y}, \check{Y}, \hat{Z}) \in (\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2)^{\mathcal{I}}$ solution to the discretely reflected BSDE:

$$\begin{cases} \hat{Y}_t &= g(X_T) \vee g(X^\pi_T) + \int_t^T f(X_u, \widetilde{Y}_u) \vee f(X^\pi_{\pi(u)}, \widetilde{Y}^\pi_{\pi(u)})\mathrm{d}u \\ & \qquad - \int_t^T \hat{Z}_u.\mathrm{d}W_u + (\hat{K}^\pi_T - \hat{K}^\pi_t) , \\ \hat{K}_t &= \sum_{r\in\Re\setminus\{0\}} \Delta\hat{K}^\pi_r \mathbf{1}_{\{r\le t\}} \text{ and } \Delta\hat{K}^\pi_t = \check{Y}_t - \hat{Y}_t , \\ \check{Y}_t &= \hat{Y}_t \mathbf{1}_{\{t\notin\Re\}} + \hat{\mathcal{P}}(X_t, X^\pi_t, \hat{Y}_t)\mathbf{1}_{\{t\in\Re\}} , \qquad 0 \le t \le T. \end{cases} \tag{4.3.10}$$

where $\hat{\mathcal{P}} : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d$ is defined by

$$\hat{\mathcal{P}}(x_1, x_2, y)^i = \max_j\{y^j - c_{i,j}(x_1) \wedge c_{i,j}(x_2)\} = \mathcal{P}(x_1, y)^i \vee \mathcal{P}(x_2, y)^i$$

Using a backward induction argument and the comparison theorem for non-reflected BSDEs, we easily obtain the following property.

Lemma 4.3.1 *We have $\hat{Y}_t \succeq \max\{\widetilde{Y}_t, \widetilde{Y}^\pi_t\}$ for all $t \in [0,T]$, $\mathbb{P}-a.s.$.*

Since $\mathcal{P}$ is only L-Lipschitz with $L > 1$, the "geometric" approach, used in the proof of Proposition 4.3.1, does not allow to retrieve convenient controls. Instead, we use here the correspondance between obliquely reflected BSDEs and optimal switching problem.

Fix $(t,i) \in [0,T] \times \mathcal{I}$. For any strategy $a = (\theta_j, \alpha_j)_{j\in\mathbb{N}} \in \mathcal{A}^{\Re}_{t,i}$, we introduce the following switched BSDEs:

$$
\begin{aligned}
\tilde U^a_u &= g^{a_T}(X_T) + \int_u^T f^{a_s}(X_s, \tilde Y_s)\mathrm{d}s - \int_u^T \tilde V^a_s.\mathrm{d}W_s - \tilde A^a_T + \tilde A^a_u\,, \quad t \le u \le T\,,\\
\tilde U^{\pi,a}_u &= g^{a_T}(X^\pi_T) + \int_u^T f^{a_s}(X^\pi_{\pi(s)}, \widetilde Y^\pi_{\pi(s)})\mathrm{d}s - \int_u^T \tilde V^{\pi,a}_s.\mathrm{d}W_s - \tilde A^{\pi,a}_T + \tilde A^{\pi,a}_u\,, \quad t \le u \le T\,,\\
\hat U^a_u &= g^{a_T}(X_T) \vee g^{a_T}(X^\pi_T) + \int_u^T f^{a_s}(X_s, \tilde Y_s) \vee f^{a_s}(X^\pi_{\pi(s)}, \widetilde Y^\pi_{\pi(s)})\mathrm{d}s\\
&\quad - \int_u^T \hat V^a_s.\mathrm{d}W_s - \hat A^a_T + \hat A^a_u\,, \quad t \le u \le T\,,
\end{aligned}
\tag{4.3.11}
$$

where

$$
\begin{aligned}
\tilde A^a_u &:= \sum_{j\in\mathbb{N}^*} c_{\alpha_{j-1},\alpha_j}(X_{\theta_j})\mathbf{1}_{\{\theta_j \le u \le T\}}\,, \quad t \le u \le T\,,\\
\tilde A^{\pi,a}_u &:= \sum_{j\in\mathbb{N}^*} c_{\alpha_{j-1},\alpha_j}(X^\pi_{\pi(\theta_j)})\mathbf{1}_{\{\theta_j \le u \le T\}}\,, \quad t \le u \le T\,,\\
\hat A^a_u &:= \sum_{j\in\mathbb{N}^*} c_{\alpha_{j-1},\alpha_j}(X_{\theta_j}) \wedge c_{\alpha_{j-1},\alpha_j}(X^\pi_{\pi(\theta_j)})\mathbf{1}_{\{\theta_j \le u \le T\}}\,, \quad t \le u \le T\,.
\end{aligned}
$$

We denote by $\hat a = (\hat\theta_i, \hat\alpha_i)_{i\ge 0} \in \mathcal{A}^{\Re}_{t,i}$ the optimal strategy starting from i at t ($\hat\theta_0 = t$ and $\hat\alpha_0 = i$) associated to $\hat Y^i_t$ recalling Theorem 4.2.1, Remark 4.2.3.

We first need to control the variable $N^{\hat a}$. To this end we control the moments of $(\hat Y, \hat Z)$ by the following proposition whose proof is postponed to the Appendix.

Proposition 4.3.2 *For $|\pi|L < 1$, the following bound holds*

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |\hat Y_t|^p\right] + \mathbb{E}\left[(\int_0^T |\hat Z_t|^2 dt)^{\frac{p}{2}}\right] \;\le\; C^p_L, \quad p \ge 2\,,
$$

recall that C^p_L neither depends on $\Re$ nor on π.

Using standard arguments, recall (4.3.11), we then get

$$
\mathbb{E}\Big[|\hat A^{\hat a}_T|^p\Big] \le C^p_L \mathbb{E}\left[\sup_{s\in[0,T]} |\hat U^{\hat a}_s|^p + (\int_0^T |\hat V^{\hat a}_s|^2 \mathrm{d}s)^{\frac{p}{2}} + |\hat A^{\hat a}_t|^p\right]
$$

By definition of $\hat a$ and (4.2.4), we have $|A^{a^*}_t| \le C_L(1 + |X_t|)$ which gives with the previous inequality and the link between $(\hat U^{\hat a}_s, \hat V^{\hat a}_s)$ and $(\hat Y^{\hat a}_s, \hat Z^{\hat a}_s)$

$$
\mathbb{E}\Big[|A^{a^*}_T|^p\Big] \le C_L
$$

From (4.2.4) we get $C_L\mathbb{E}\big[|N^{a^*}|^p\big] \leq \mathbb{E}\big[|A_T^{a^*}|^p\big]$ which completes the proof.

$$\mathbb{E}\Big[|\hat{A}_T^{\hat{a}}|^p + |N^{\hat{a}}|^p\Big] \leq C_L^p\,. \tag{4.3.12}$$

Proof of Theorem 4.3.1.

Step 1. We first prove that the following holds

$$\mathbb{E}|\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\hat{a}}|^2 + \mathbb{E}|\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\pi,\hat{a}}|^2 \quad \leq \quad C_L|\pi|\,, \quad t \leq u \leq T\,. \tag{4.3.13}$$

Indeed, using the inequality $|x \wedge y - y| \leq |x - y|$ for $x, y \in \mathbb{R}$ and the Lipschitz property of $c_{i,j}$, we compute

$$\begin{aligned}
\mathbb{E}|\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\hat{a}}|^2 + \mathbb{E}|\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\pi,\hat{a}}|^2 \quad &= \quad \mathbb{E}\Big|\sum_{k=1}^{N^{\hat{a}}}[c_{\hat{\alpha}_{k-1},\hat{\alpha}_k}(X_{\hat{\theta}_k}) \wedge c_{\hat{\alpha}_{k-1},\hat{\alpha}_k}(X^\pi_{\hat{\theta}_k}) - c_{\hat{\alpha}_{k-1},\hat{\alpha}_k}(\hat{X}_{\hat{\theta}_k})]\mathbf{1}_{\hat{\theta}_k \leq u}\Big|^2 \\
&\quad +\mathbb{E}\Big|\sum_{k=1}^{N^{\hat{a}}}[c_{\hat{\alpha}_{k-1},\hat{\alpha}_k}(X_{\hat{\theta}_k}) \wedge c_{\hat{\alpha}_{k-1},\hat{\alpha}_k}(X^\pi_{\hat{\theta}_k}) - c_{\hat{\alpha}_{k-1},\hat{\alpha}_k}(X^\pi_{\hat{\theta}_k})]\mathbf{1}_{\hat{\theta}_k \leq u}\Big|^2 \\
&\leq \quad C_L\mathbb{E}[|N^{\hat{a}}|^2 \sup_{s\in[0,T]}|X_s - X_s^\pi|^2]\,.
\end{aligned}$$

Using the Cauchy-Schwartz inequality and standard estimates (4.3.3) on the discretization of diffusions we get

$$\mathbb{E}|\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\hat{a}}|^2 + \mathbb{E}|\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\pi,\hat{a}}|^2 \quad \leq \quad C_L|\pi|\sqrt{\mathbb{E}[|N^{\hat{a}}|^4]}\,,$$

which combined with (4.3.12) leads to the inequality (4.3.13).

Step 2. Define for $u \in [t,T]$

$$\delta U_u^{\hat{a}} \quad = \quad \hat{U}_u^{\hat{a}} - \tilde{U}_u^{\hat{a}}, \quad \delta U_u^{\pi,\hat{a}} \quad = \quad \hat{U}_u^{\hat{a}} - \tilde{U}_u^{\pi,\hat{a}}\,,$$

and

$$\delta\Gamma_u^{\hat{a}} \quad = \quad \delta U_u^{\hat{a}} - (\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\hat{a}}), \quad \delta\Gamma_u^{\pi,\hat{a}} \quad = \quad \delta U_u^{\pi,\hat{a}} - (\hat{A}_u^{\hat{a}} - \tilde{A}_u^{\pi,\hat{a}})\,.$$

Applying Itô 's formula, we have

$$\begin{aligned}
\mathbb{E}|\delta\Gamma_u^{\hat{a}}|^2 \quad &\leq \quad \mathbb{E}|g^{\hat{a}_T}(X_T) \vee g^{\hat{a}_T}(X_T^\pi) - g^{\hat{a}_T}(X_T)|^2 \\
&\quad +2\mathbb{E}\int_u^T \delta U_s^{\hat{a}}[f^{\hat{a}_s}(X_s,\widetilde{Y}_s) \vee f^{\hat{a}_s}(X^\pi_{\pi(s)},\widetilde{Y}^\pi_{\pi(s)}) - f^{\hat{a}_s}(X_s,\widetilde{Y}_s)]\mathrm{d}s\,, \quad t \leq u \leq T\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}|\delta\Gamma_u^{\pi,a}|^2 \quad &\leq \quad \mathbb{E}|g^{\hat{a}_T}(X_T^\pi) \vee g^{\hat{a}_T}(X_T) - g^{\hat{a}_T}(X_T^\pi)|^2 \\
&\quad +2\mathbb{E}\int_u^T \delta U_s^{\pi,\hat{a}}[f^{\hat{a}_s}(X_s,\widetilde{Y}_s) \vee f^{\hat{a}_s}(X^\pi_{\pi(s)},\widetilde{Y}^\pi_{\pi(s)}) - f^{\hat{a}_s}(X^\pi_{\pi(s)},\widetilde{Y}^\pi_{\pi(s)})]\mathrm{d}s\,, \quad t \leq u \leq T\,.
\end{aligned}$$

Using the inequalities $|x \vee y - y| \leq |x - y|$ and $2xy \leq |x|^2 + |y|^2$ for all $x, y \in \mathbb{R}$, and the Lipshitz property of f and g we get

$$\mathbb{E}\Big[|\delta\Gamma_u^{\hat{a}}|^2 + |\delta\Gamma_u^{\pi,\hat{a}}|^2\Big] \leq C_L\Big(\mathbb{E}\Big[\sup_{s\in[0,T]}|X_{\pi(s)} - X^\pi_{\pi(s)}|^2\Big] + \sup_{s\in[0,T]}\mathbb{E}\Big[\sup_{u\in[0,T],|u-s|\leq|\pi|}|X_s - X_u|^2\Big] + \int_u^T \mathbb{E}\Big[|\delta U_s^{\hat{a}}|^2 + |\delta U_s^{\pi,\hat{a}}|^2\Big]\mathrm{d}s + \int_u^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}^\pi_{\pi(s)}|^2\mathrm{d}s\Big), \quad t \leq u \leq T.$$

From standard estimates on diffusions (4.2.2)-(4.3.3) we have

$$\mathbb{E}\Big[|\delta\Gamma_u^{\hat{a}}|^2 + |\delta\Gamma_u^{\pi,\hat{a}}|^2\Big] \leq C_L\Big(|\pi| + \int_u^T \mathbb{E}\Big[|\delta U_s^{\hat{a}}|^2 + |\delta U_s^{\pi,\hat{a}}|^2\Big]\mathrm{d}s + \int_u^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}^\pi_{\pi(s)}|^2\mathrm{d}s\Big), \quad t \leq s \leq T. \tag{4.3.14}$$

By definition of $\Gamma^{\hat{a}}$ and $\Gamma^{\pi,\hat{a}}$, (4.3.13) and (4.3.14) we obtain

$$\mathbb{E}\Big[|\delta U_u^{\hat{a}}|^2 + |\delta U_u^{\pi,\hat{a}}|^2\Big] \leq C_L\Big(|\pi| + \int_u^T \mathbb{E}\Big[|\delta U_s^{\hat{a}}|^2 + |\delta U_s^{\pi,\hat{a}}|^2\Big]ds + \int_u^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}^\pi_{\pi(s)}|^2\mathrm{d}s\Big), \quad t \leq u \leq T.$$

Applying Gronwall's Lemma we get

$$\mathbb{E}\Big[|\delta U_t^{\hat{a}}|^2 + |\delta U_t^{\pi,\hat{a}}|^2\Big] \leq C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}^\pi_{\pi(s)}|^2\mathrm{d}s\Big). \tag{4.3.15}$$

Then, by definition of $\hat{a}$ and Lemma 4.3.1, for any strategy $a \in \mathcal{A}^{\Re}_{t,i}$, we have (see Theorem 4.2.1)

$$\tilde{U}_t^{\pi,a} \leq \widetilde{Y}_t^{\pi,i} \leq \hat{Y}_t^i = \hat{U}_t^{\hat{a}} \qquad \text{and} \qquad \tilde{U}_t^a \leq \widetilde{Y}_t^i \leq \hat{Y}_t^i = \hat{U}_t^{\hat{a}},$$

which gives

$$\mathbb{E}|\widetilde{Y}_t^i - \widetilde{Y}_t^{\pi,i}|^2 \leq C_L\mathbb{E}\Big[|\delta U_t^{\hat{a}}|^2 + |\delta U_t^{\pi,\hat{a}}|^2\Big]. \tag{4.3.16}$$

Combining (4.3.16) and (4.3.15), we have

$$\mathbb{E}|\widetilde{Y}_t^i - \widetilde{Y}_t^{\pi,i}|^2 \leq C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}_{\pi(s)}|^2 ds + \int_t^T \mathbb{E}|\widetilde{Y}^\pi_{\pi(s)} - \widetilde{Y}_{\pi(s)}|^2\mathrm{d}s\Big).$$

Since i is arbitrary chosen, we get

$$\mathbb{E}|\widetilde{Y}_t - \widetilde{Y}_t^{\pi}|^2 \leq C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}_{\pi(s)}|^2 ds + \int_t^T \mathbb{E}|\widetilde{Y}^\pi_{\pi(s)} - \widetilde{Y}_{\pi(s)}|^2\mathrm{d}s\Big). \tag{4.3.17}$$

The same reasoning applied at time $t_j \in \pi$ with $t_j \geq t$ leads to

$$\mathbb{E}|\widetilde{Y}_{t_j} - \widetilde{Y}_{t_j}^{\pi}|^2 \leq C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}_{\pi(s)}|^2 ds + |\pi|\sum_{k=j+1}^n \mathbb{E}|\widetilde{Y}^\pi_{t_k} - \widetilde{Y}_{t_k}|^2\Big).$$

Applying the discrete version of Gronwall's lemma, we deduce

$$\mathbb{E}|\widetilde{Y}_{t_j} - \widetilde{Y}^{\pi}_{t_j}|^2 \;\; \leq \;\; C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}_{\pi(s)}|^2 ds\Big)\;, \qquad \text{for} \;\; t_j \geq t, \quad t_j \in \pi.$$

Plugging this estimate into (4.3.17), we compute

$$\mathbb{E}|\widetilde{Y}_t - \widetilde{Y}^{\pi}_t|^2 \;\; \leq \;\; C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s - \widetilde{Y}_{\pi(s)}|^2 ds\Big)\;.$$

The proof is concluded using the regularity of $\widetilde{Y}$ given by Lemma 4.3.2 bellow. □

Lemma 4.3.2 *When f does not depend on z, the following holds for the discretely reflected BSDE,*

$$\mathbb{E}\Big[\int_0^T |\widetilde{Y}_t - \widetilde{Y}_{\pi(t)}|^2 \mathrm{d}t\Big] \leq C_L|\pi|.$$

Proof. The process (Y, Z) is the solution of a discretely reflected BSDE. It follows then from Proposition 4.2.1, that

$$\mathbb{E}\Big[\sup_{s\in[0,T]} |\widetilde{Y}_s|^2 + \int_0^T |Z_s|^2 \mathrm{d}s\Big] \leq C_L. \tag{4.3.18}$$

For $t_i \leq t < t_{i+1}$, $i \in \{0, \ldots, n-1\}$, we have

$$\widetilde{Y}_{t_i} - \widetilde{Y}_t = \int_{t_i}^t f(X_{t_i}, \widetilde{Y}_{t_i})\mathrm{d}s - \int_{t_i}^t Z_s.\mathrm{d}W_s\;,$$

and we easily compute

$$\mathbb{E}\Big[|\widetilde{Y}_{t_i} - \widetilde{Y}_t|^2\Big] \leq C_L\left((1 + \mathbb{E}[|X_{t_i}|^2 + |Y_{t_i}|^2])|\pi| + \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s|^2 \mathrm{d}s\Big]\right)\;.$$

Integrating on $[t_i, t_{i+1})$ and summing on i leads to

$$\mathbb{E}\Big[\int_0^T |\widetilde{Y}_t - \widetilde{Y}_{\pi(t)}|^2 \mathrm{d}t\Big] \leq C_L|\pi|(1 + \sup_{1\leq i\leq n} \mathbb{E}\Big[|X_{t_i}|^2 + |\widetilde{Y}_{t_i}|^2\Big] + \mathbb{E}\Big[\int_0^T |Z_s|^2 \mathrm{d}s\Big]).$$

The proof is then concluded combining the last inequality with (4.3.18) and (4.3.2). □

4.4 Extension to the continuously reflected case

In this section, we extend the convergence results of the scheme (4.3.4) to the case of continuously reflected BSDEs. To this end, we show that the error between discretely and continuously reflected BSDEs is controled in a convenient way.

4.4.1 Convergence to continuously obliquely reflected BSDEs

In the sequel, we shall use the following assumption on f:

($\mathbf{H}f$) The function f is bounded in its last variable : $\sup_{z\in\mathcal{M}^{d,d}}|f(0,0,z)|\leq C_L$.

We denote by $(\dot Y,\dot Z,\dot K)\in(\mathcal{S}^{\mathbf{2}}\times\mathcal{H}^{\mathbf{2}}\times\mathbf{A}^{\mathbf{2}})^{\mathcal{I}}$ the solution of the continuously obliquely reflected BSDE:

$$\begin{cases}\dot Y^i_t=g^i(X_T)+\int_t^T f^i(X_s,\dot Y^i_s,\dot Z^i_s)\mathrm{d}s-\int_t^T\dot Z^i_s.\mathrm{d}W_s+\dot K^i_T-\dot K^i_t\;,\\ \dot Y^i_t\geq\max_{j\in\mathcal{I}}\{\dot Y^j_t-c_{i,j}(X_t)\}\;,\quad 0\leq t\leq T\;,\\ \int_0^T[\dot Y^i_t-\max_{j\in\mathcal{I}}\{\dot Y^j_t-c_{i,j}(X_t)\}]\mathrm{d}\dot K^i_t=0\;,\quad i\in\mathcal{I}\;.\end{cases}\tag{4.4.1}$$

Under the assumption on f, g and c, the existence and uniqueness of such a solution is given by [41, 44]. As in the discretely reflected case, the process $\dot Y^i$, $i\in\mathcal{I}$ can be interpreted (see Theorem 3.1 in [44]) as the the Snell envelope of the family of processes $(U^a)_a$, recall (4.2.9):

$$\dot Y^i_t\;=\;\operatorname*{ess\,sup}_{a\in\mathcal{A}_{t,i}}U^a_t.$$

It follows then from (4.2.12) that

$$\dot Y\succeq Y\succeq\widetilde Y\;,\tag{4.4.2}$$

for any grid $\Re$.

According to the proof of Theorem 3.1 in Hu and Tang [44], there exists, for each initial condition $(t_0,i_0)\in[0,T]\times\mathcal{I}$, an optimal switching strategy $\dot a:=(\dot\theta_k,\dot\alpha_k)_{k\geq0}\in\mathcal{A}_{t_0,i_0}$, such that $\dot Y^{i_0}_{t_0}=U^{\dot a}_{t_0}$. Moreover we have $\mathbb{E}|A^{\dot a}_T|^p<\infty$ for all $p\geq1$. Indeed, since $(\dot Y^{\dot a},\dot Z^{\dot a})$ satisfy the switched equation (4.2.9) with $a=\dot a$ (see Theorem 3.1 in [44]) and $(\dot Y,\dot Z)\in(\mathcal{S}^{\mathbf{p}}\times\mathcal{H}^{\mathbf{p}})^{\mathcal{I}}$ (see Proposition 4.5.2 in the Appendix), we have

$$\mathbb{E}\Big[\sup_{s\in[t_0,T]}|U^{\dot a}_s|^p\Big]+\mathbb{E}\Big[(\int_{t_0}^T|V^{\dot a}_s|^2ds)^{\frac{p}{2}}\Big]\leq C^p_L\;,$$

Then, writing the equation satisfied by $(U^{\dot a},V^{\dot a})$ and using standard arguments for BSDEs, we get

$$\mathbb{E}\big[|A^{\dot a}_T|^p\big]\leq C_L(\mathbb{E}\Big[\sup_{s\in[t_0,T]}|U^{\dot a}_s|^p\Big]+\mathbb{E}\Big[(\int_{t_0}^T|V^{\dot a}_s|^2ds)^{\frac{p}{2}}\Big]+\mathbb{E}[|A^{\dot a}_t|^p])$$

By definition of $\dot a$ (see Theorem 3.1 in [44]) and (4.2.4), we have $|A^{\dot a}_t|\leq C_L(1+|X_t|)$ which gives with the previous inequality

$$\mathbb{E}\big[|A^{\dot a}_T|^p+|N^{\dot a}|^p\big]\;\leq\;C^p_L\,.\tag{4.4.3}$$

We now present a key result in the proof of the convergence of the solutions of discretely reflected BSDEs to the solutions of the corresponding continuously reflected ones.

Theorem 4.4.1 *Under* $(\mathbf{H}f)$*, the following holds*

$$\mathbb{E}\Big[\sup_{r\in\Re}\Big\{|\dot{Y}_r-\widetilde{Y}_r|^2+|\dot{Y}_r-Y_r|^2\Big\}\Big] \;\leq\; C_L^{\varepsilon}|\Re|^{1-\varepsilon}\;. \tag{4.4.4}$$

for all $\varepsilon>0$*. Moreover if the cost functions are constant we have*

$$\mathbb{E}\Big[\sup_{r\in\Re}\Big\{|\dot{Y}_r-\widetilde{Y}_r|^2+|\dot{Y}_r-Y_r|^2\Big\}\Big] \;\leq\; C_L|\Re|\;. \tag{4.4.5}$$

The proof of this result relies mainly on the interpretation in terms of switched BSDEs provided in Section 4.2.2.

To an optimal strategy $\dot{a}=(\dot{\theta}_k,\dot{\alpha}_k)_k\in\mathcal{A}_{t_0,i_0}$ not restricted to lie in the grid $\Re$, we associate the corresponding 'discretized' strategy $a:=(\theta_k,\alpha_k)_{k\geq 0}\in\mathcal{A}^{\Re}_{t_0,i_0}$ defined by

$$\theta_k \;:=\; \inf\Big\{r\geq\dot{\theta}_k\;;\;\; r\in\Re\Big\} \qquad\text{and}\qquad \alpha_k:=\dot{\alpha}_k\;,\qquad k\geq 0\;. \tag{4.4.6}$$

Proof. Step 1. We first prove two key controls of the error between $A^{\dot{a}}$ and A^{a}.

We compute for $h\geq 2$

$$\begin{aligned}\Big(\int_{t_0}^T|A_s^{\dot{a}}-A_s^{a}|^2\mathrm{d}s\Big)^{\frac{h}{2}} \;&=\; \Bigg(\int_{t_0}^T\Bigg|\sum_{k=1}^{N^{\dot{a}}}c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\dot{\theta}_k})\mathbf{1}_{\dot{\theta}_k\leq s}-c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\theta_k})\mathbf{1}_{\theta_k\leq s}\Bigg|^2\mathrm{d}s\Bigg)^{\frac{h}{2}}\\ &\leq\; C_L^h\Bigg(\int_{t_0}^T\Bigg|\sum_{k=1}^{N^{\dot{a}}}c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\dot{\theta}_k})\mathbf{1}_{\dot{\theta}_k\leq s<\theta_k}\Bigg|^2\mathrm{d}s\Bigg)^{\frac{h}{2}} \qquad (4.4.7)\\ &\quad+C_L^h\int_{t_0}^T\Bigg|\sum_{k=1}^{N^{\dot{a}}}\Big[c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\theta_k})-c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\dot{\theta}_k})\Big]\mathbf{1}_{\theta_k\leq s}\Bigg|^h\mathrm{d}s\;.\end{aligned}$$

Using the convexity inequality $(\sum_{k=1}^n|x_k|)^2\leq n\sum_{k=1}^n|x_k|^2$, we obtain

$$\Bigg(\int_{t_0}^T\Bigg|\sum_{k=1}^{N^{\dot{a}}}c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\dot{\theta}_k})\mathbf{1}_{\dot{\theta}_k\leq s<\theta_k}\Bigg|^2\mathrm{d}s\Bigg)^{\frac{h}{2}} \;\leq\; C_L^h(1+\sup_{t\in[0,T]}|X_t|^h)|N^{\dot{a}}|^h|\Re|^{\frac{h}{2}} \tag{4.4.8}$$

Then, from the Lipschitz property of the maps $c_{i,j}$ and the convexity inequality $(\sum_{k=1}^n|x_k|)^h$ $\leq n^{h-1}\sum_{k=1}^n|x_k|^h$, we get

$$\int_{t_0}^T\Bigg|\sum_{k=1}^{N^{\dot{a}}}\Big[c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\theta_k})-c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\dot{\theta}_k})\Big]\mathbf{1}_{\theta_k\leq s}\Bigg|^h\mathrm{d}s \;\leq\; C_L^h|N^{\dot{a}}|^{h-1}\int_{t_0}^T\sum_{k=1}^{N^{\dot{a}}}|X_{\theta_k}-X_{\dot{\theta}_k}|^h\mathbf{1}_{\theta_k\leq s}\mathrm{d}s$$

By definition of $\dot{\theta}_k$ and θ_k, we get

$$\int_{t_0}^{T} \left| \sum_{k=1}^{N^{\dot{a}}} \left[c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\theta_k}) - c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\dot{\theta}_k}) \right] \mathbf{1}_{\theta_k \leq s} \right|^h \mathrm{d}s \quad \leq$$

$$C_L^h |N^{\dot{a}}|^h \int_{t_0}^{T} \sum_{k=1}^{\kappa} \sup_{r \in [r_{k-1}, r_k]} |X_r - X_{r_k}|^h \mathbf{1}_{\theta_k \leq s} \mathrm{d}s \tag{4.4.9}$$

Combining (4.4.7) with (4.4.8) and (4.4.9) leads to

$$\left(\int_{t_0}^{T} \left| A_s^{\dot{a}} - A_s^a \right|^2 \mathrm{d}s \right)^{\frac{h}{2}} \;\leq\; C_L^h |N^{\dot{a}}|^h \left(1 + \sup_{s \in [0,T]} |X_s|^h |\Re|^{\frac{h}{2}} + \chi^{|\Re|,h} \right) . \tag{4.4.10}$$

where $\chi^{|\Re|,h} := \sum_{k=1}^{\kappa} \sup_{r \in [r_{k-1}, r_k]} |X_r - X_{r_k}|^h$.

For $r \in \Re$, we have $\mathbf{1}_{\dot{\theta}_k \leq r} = \mathbf{1}_{\theta_k \leq r}$ which gives

$$|A_r^{\dot{a}} - A_r^a|^h \;\leq\; (\sum_{k=1}^{N^{\dot{a}}} \left| c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\dot{\theta}_k}) - c_{\dot{\alpha}_{k-1},\dot{\alpha}_k}(X_{\theta_k}) \right| \mathbf{1}_{\theta_k \leq r})^h \;\leq\; C_L^h |N^{\dot{a}}|^h \chi^{|\Re|,h} . \tag{4.4.11}$$

Step 2. Recall that (t_0, i_0) is given. Let us introduce the processes $\Gamma := U^a - A$ and $\dot{\Gamma} := U^{\dot{a}} - A^{\dot{a}}$, so that the dynamics of $\dot{\Gamma} - \Gamma$ on $[t_0, T]$ is given by

$$\dot{\Gamma}_t - \Gamma_t \;=\; \int_t^T \left[f^{\dot{a}_s}(X_s, \dot{\Gamma}_s + A_s^{\dot{a}}, V_s^{\dot{a}}) - f^{a_s}(X_s, \Gamma_s + A_s^a, V_s^a) \right] \mathrm{d}s - \int_t^T (V_s^{\dot{a}} - V_s^a).\mathrm{d}W_s .$$

Observe that,

$$|U^a - U^{\dot{a}}| \;\leq\; |\Gamma - \dot{\Gamma}| + |A^a - A^{\dot{a}}| . \tag{4.4.12}$$

Applying Ito's formula to the continuous process $|\dot{\Gamma} - \Gamma|^2$ on $[t_0, T]$, using Gronwall Lemma and the Lipschitz property of f, we obtain

$$\begin{aligned} |\dot{\Gamma}_{t_0} - \Gamma_{t_0}|^2 \;\leq\; C_L \mathbb{E}\Big[& \int_{t_0}^{T} \left| f^{\dot{a}_s}(X_s, U_s^{\dot{a}}, V_s^{\dot{a}}) - f^{a_s}(X_s, U_s^{\dot{a}}, V_s^{\dot{a}}) \right|^2 \mathrm{d}s \\ & + \int_{t_0}^{T} \left| A_s^{\dot{a}} - A_s^a \right|^2 \mathrm{d}s \Big| \mathcal{F}_{t_0} \Big] , \end{aligned} \tag{4.4.13}$$

up to power $\frac{h}{2}$, we get from Jensen inequality

$$\begin{aligned} |\dot{\Gamma}_{t_0} - \Gamma_{t_0}|^h \;\leq\; C_L^h \mathbb{E}\Big[& \left(\int_{t_0}^{T} \left| f^{\dot{a}_s}(X_s, U_s^{\dot{a}}, V_s^{\dot{a}}) - f^{a_s}(X_s, U_s^{\dot{a}}, V_s^{\dot{a}}) \right|^2 \mathrm{d}s \right)^{\frac{h}{2}} \\ & + \left(\int_{t_0}^{T} \left| A_s^{\dot{a}} - A_s^a \right|^2 \mathrm{d}s \right)^{\frac{h}{2}} \Big| \mathcal{F}_{t_0} \Big] , \end{aligned} \tag{4.4.14}$$

Under $(\mathbf{H}f)$, we compute that

$$\begin{aligned}
\int_{t_0}^{T}\left|f^{\dot{a}_s}(X_s,U_s^{\dot{a}},V_s^{\dot{a}})-f^{a_s}(X_s,U_s^{\dot{a}},V_s^{\dot{a}})\right|^2\mathrm{d}s &= \\
\int_{t_0}^{T}\left|\sum_{k=1}^{N^{\dot{a}}} f^{\alpha_{k-1}}(X_s,U_s^{\dot{a}},V_s^{\dot{a}})\left(\mathbf{1}_{\dot{\theta}_{k-1}\le s<\dot{\theta}_k}-\mathbf{1}_{\theta_{k-1}\le s<\theta_k}\right)\right|^2\mathrm{d}s &\le \\
C_L|N^{\dot{a}}|\sum_{k=1}^{N^{\dot{a}}}\int_{t_0}^{T}(1+|X_s|^2+|U_s^{\dot{a}}|^2)\left|\mathbf{1}_{\dot{\theta}_{k-1}\le s<\dot{\theta}_k}-\mathbf{1}_{\theta_{k-1}\le s<\theta_k}\right|\mathrm{d}s &
\end{aligned}$$

which leads to

$$\left(\int_{t_0}^{T}\left|f^{\dot{a}_s}(X_s,U_s^{\dot{a}},V_s^{\dot{a}})-f^{a_s}(X_s,U_s^{\dot{a}},V_s^{\dot{a}})\right|^4\mathrm{d}s\right)^{\frac{h}{2}} \le C_L^h|N^{\dot{a}}|^h\sup\nolimits_{s\in[0,T]}(1+|X_s|^h+|U_s^{\dot{a}}|^h)|\Re|^{\frac{h}{2}}$$

Combining the last inequality with (4.4.10), (4.4.11), (4.4.12) and (4.4.14), we then obtain

$$|\dot{Y}_{t_0}^{i_0}-\widetilde{Y}_{t_0}^{i_0}|^h \le C_L^h\mathbb{E}\left[|N^{\dot{a}}|^h\left(\sup_{t\in[0,T]}(1+|X_s|^h+|U_s^{\dot{a}}|^h)|\Re|^{\frac{h}{2}}+(\chi^{|\Re|,h})\right)\Big|\mathcal{F}_{t_0}\right].$$

Since i_0 is arbitrary chosen we get

$$\mathbb{E}\left[\sup_{r\in\Re}|\dot{Y}_r-\widetilde{Y}_r|^2\right] \le C_L^h\left(\mathbb{E}\left[\sup_{t\in[0,T]}|M_t^1|^{\frac{h}{2}}\right]|\Re|+\mathbb{E}\left[\sup_{t\in[0,T]}|M_t^2|^{\frac{h}{2}}\right]\right), \quad (4.4.15)$$

where M^1 and M^2 are the martingales defined by

$$\begin{aligned}
M_t^1 &= \mathbb{E}\left[|N^{\dot{a}}|^h\sup_{s\in[0,T]}(1+|X_s|^h+|U_s^{\dot{a}}|^h)\Big|\mathcal{F}_t\right],\\
M_t^2 &= \mathbb{E}\left[|N^{\dot{a}}|^h\chi^{|\Re|,h}\Big|\mathcal{F}_t\right], \qquad 0\le t\le T.
\end{aligned}$$

From Burkholder-Davis-Gundy inequality we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}|M_t^1|^{\frac{h}{2}}\right] \le C_L^h. \quad (4.4.16)$$

Still using Burkholder-Davis-Gundy inequality we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}|M_t^2|^{\frac{h}{2}}\right] \le C_L^h\left(|M_0^2|^{\frac{h}{2}}+\mathbb{E}\left[|M_T|^2\right]^{\frac{1}{h}}\right).$$

From Cauchy-Schwarz and Jensen inequalities we get

$$\mathbb{E}\left[\sup_{t\in[0,T]}|M_t^2|^{\frac{h}{2}}\right] \le C_L^h\left(|M_0^2|^{\frac{h}{2}}+\mathbb{E}\left[|\chi^{|\Re|,h}|^2\right]^{\frac{1}{h}}\right).$$

Using then (4.2.2) the convexity inequality $(\sum_{k=1}^n |x^k|)^p \leq n^{p-1}\sum_{k=1}^n |x^k|^p$ for $p \geq 1$, and the condition $\kappa|\Re| \leq L$, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M_t^2|^{\frac{h}{2}}\right] \leq C_L^h |\Re|^{1-\frac{2}{h}} .$$

Combining this last inequality with (4.4.15), (4.4.16) and the Lipschitz property of $\mathcal{P}$ (see Lemma 4.2.1), we deduce (4.4.4).

We now prove (4.4.5). Suppose that the cost functions are constant. Then the same arguments as in Step 1. give

$$\int_{t_0}^{T} |A_s^{\dot{a}} - A_s^{\dot{a}}|^2 \mathrm{d}s \leq C_L|\Re|$$

and $A_r^{\dot{a}} - A_r^{a} = 0$ for $r \in \Re$. This gives with (4.4.13)

$$|\dot{Y}_{t_0}^{i_0} - \widetilde{Y}_{t_0}^{i_0}|^2 \leq C_L|\Re|\mathbb{E}\left[|N^{\dot{a}}|^2 \sup_{t\in[0,T]}(1+|X_s|^2+|U_s^{\dot{a}}|^2)\Big|\mathcal{F}_{t_0}\right] . \quad (4.4.17)$$

Using Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}\Big[\sup_{t\in\Re} |\dot{Y}_t - \widetilde{Y}_t|^2\Big] \leq C_L|\Re| .$$

Combined the Lipschitz property of $\mathcal{P}$, we get (4.4.5). □

We now give a control of the error on the variable Z in the case where the maps $c_{i,j}$ are constant.

Theorem 4.4.2 *If the cost functions are constant, the following holds under* $(\mathbf{H}f)$

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\dot{Y}_t - \widetilde{Y}_t|^2 + |\dot{Y}_t - Y_t|^2 + \int_t^T |\dot{Z}_s - Z_s|^2 \mathrm{d}s\Big] \leq C_L|\Re|^{\frac{1}{2}} .$$

Proof.

Step 1.a. Fix $t \in [0,T]$. We introduce $\delta\widetilde{Y} = \dot{Y} - \widetilde{Y}$, $\delta Y = \dot{Y} - Y$, $\delta Z = \dot{Z} - Z$ and $\delta f = f(X,\dot{Y},\dot{Z}) - f(X,\widetilde{Y},Z)$. Applying Ito's formula to the *càdlàg* process $|\delta\widetilde{Y}|^2$, we obtain

$$\begin{aligned} |\delta\widetilde{Y}_t|^2 + \int_t^T |\delta Z_s|^2 \mathrm{d}s &= |\delta\widetilde{Y}_T|^2 - 2\int_{(t,T]} \delta\widetilde{Y}_{s-}.\mathrm{d}\delta\widetilde{Y}_s \\ &\quad - \sum_{t<s\leq T} (|\delta\widetilde{Y}_s|^2 - |\delta\widetilde{Y}_{s-}|^2 - 2\delta\widetilde{Y}_{s-}.\Delta\delta\widetilde{Y}_s) . \end{aligned} \quad (4.4.18)$$

Recall $\delta\widetilde{Y}_{s-} = \delta Y_s$ for $s \in [0,T]$. Since

$$\sum_{t<s\leq T} (|\delta\widetilde{Y}_s|^2 - |\delta\widetilde{Y}_{s-}|^2 - 2\delta\widetilde{Y}_{s-}.\Delta\delta\widetilde{Y}_s) = \sum_{t<s\leq T} |\delta\widetilde{Y}_s - \delta Y_s|^2 \geq 0 \quad \text{and} \quad \int_{(t,T]} \delta Y_s.\mathrm{d}\widetilde{K}_s \geq 0 ,$$

we have

$$\mathbb{E}\Big[|\delta\widetilde{Y}_t|^2 + \int_t^T |\delta Z_s|^2 \mathrm{d}s\Big] \;\; \leq \;\; \mathbb{E}\Big[|\delta\widetilde{Y}_T|^2 + 2\int_t^T \delta\widetilde{Y}_s \delta f_s \mathrm{d}s + 2\int_t^T \delta Y_s . \mathrm{d}\dot{K}_s\Big] \; .$$

Using standard arguments, we then compute

$$\mathbb{E}\Big[|\delta\widetilde{Y}_t|^2 + \int_t^T |\delta Z_s|^2 \mathrm{d}s\Big] \;\; \leq \;\; C_L \mathbb{E}\Bigg[\sum_{j=0}^{\kappa-1} \int_{r_j}^{r_{j+1}} \delta Y_s . \mathrm{d}\dot{K}_s\Bigg] \; , \tag{4.4.19}$$

recall $\Re = \{r_0, \ldots, r_\kappa\}$.

Step 1.b. We introduce (Y^d, Z^d) the solution of the following piecewise constant BSDE, for $j < \kappa$, $s \in [r_j, r_{j+1})$

$$Y^d_s \;\; = \;\; \dot{Y}_{r_{j+1}} + \int_s^{r_{j+1}} h(X_u, Y^d_u, Z^d_u) \mathrm{d}u - \int_s^{r_{j+1}} Z^d_u . \mathrm{d}W_u \; ,$$

where for all $1 \leq k \leq d$, $h^k : \mathbb{R}^m \times \mathbb{R} \times \mathcal{M}^{1,d} \to \mathbb{R}$ is defined by

$$h^k(x, y, z) \;\; := \;\; \sum_{1 \leq i \leq d} \Big(|f^i(x, y, z)| + \max_{j \neq i} |f^i(x, y + c_{ij}, z)| \Big) \; .$$

For $i \leq d$, the process $((Y^d)^i, (Z^d)^i)$ is the solution of a discretely reflected BSDE with lower barrier $\dot{Y}^i$. Using the same argument as in the proof of Proposition 4.2.1, we obtain

$$\mathbb{E}\Big[\sup_{s \in [0,T]} |Y^d_s|^p + (\int_0^T |Z^d_s|^2 \mathrm{d}s)^{\frac{p}{2}}\Big] \;\; \leq \;\; C^p_L \; , \quad p \geq 2 \; .$$

The proof of the following result is postponed to Step 2 below

$$Y^d \;\; \succeq \;\; \bar{Y}. \tag{4.4.20}$$

We then have

$$\mathbb{E}\Bigg[\sum_{j=0}^{\kappa-1} \int_{r_j}^{r_{j+1}} \delta Y_s . \mathrm{d}\dot{K}_s\Bigg] \;\; \leq \;\; \mathbb{E}\Bigg[\sum_{j=0}^{\kappa-1} \int_{r_j}^{r_{j+1}} (Y^d_s - \widetilde{Y}_s) . \mathrm{d}\dot{K}_s\Bigg] \; ,$$

and

$$\begin{aligned} \mathbb{E}\Big[\int_{r_j}^{r_{j+1}} (Y^d_s - \widetilde{Y}_s) . \mathrm{d}\dot{K}_s\Big] \;\; = \;\; & \mathbb{E}\Big[\int_{r_j}^{r_{j+1}} \delta Y_{r_{j+1}} . \mathrm{d}\dot{K}_s\Big] \\ & + \mathbb{E}\Big[\int_{r_j}^{r_{j+1}} \int_s^{r_{j+1}} (h(X_u, Y^d_u, Z^d_u) - f(X_u, \widetilde{Y}_u, \widetilde{Z}_u)) \mathrm{d}u . \mathrm{d}\dot{K}_s\Big] \; . \end{aligned} \tag{4.4.21}$$

We compute that

$$\mathbb{E}\Bigg[\sum_{j=0}^{\kappa-1} \int_{r_j}^{r_{j+1}} \delta Y_{r_{j+1}} . \mathrm{d}\dot{K}_s\Bigg] = \mathbb{E}\Bigg[\sum_{j=0}^{\kappa-1} \delta Y_{r_{j+1}} . (\dot{K}_{r_{j+1}} - \dot{K}_{r_j})\Bigg] \;\; \leq \;\; C_L \mathbb{E}\Big[\sup_{r \in \Re} |\delta Y_r| |\dot{K}_T|\Big] \; , \tag{4.4.22}$$

Observe that the function h satisfies $(\mathbf{H}f)$. Hence, we deduce from (4.4.21) and (4.4.22) that

$$\mathbb{E}\Big[\int_0^T (Y_s^d - \widetilde{Y}_s).\mathrm{d}\dot{K}_s\Big] \quad \leq \quad C_L\left(\mathbb{E}\Big[\sup_{r\in\Re}|\delta Y_r||\dot{K}_T|\Big] + |\Re|\mathbb{E}\Big[\sup_{s\in[0,T]}(1+|X_s|+|Y_s^d|+|\widetilde{Y}_s|)|\dot{K}_T|\Big]\right).$$

Using Cauchy-Schwartz inequality and recalling Proposition 4.2.1 and Theorem 4.4.1, we deduce

$$\mathbb{E}\Big[\int_0^T (Y_s^d - \widetilde{Y}_s).\mathrm{d}\dot{K}_s\Big] \quad \leq \quad C_L|\Re|^{\frac{1}{2}} . \tag{4.4.23}$$

Together with (4.4.19), this estimate provides the announced result.

Step 2. We now prove (4.4.20).

As in [41], we consider the following sequence of multidimensional reflected BSDEs, for $n \geq 0$,

$$\begin{aligned}
(Y_t^n)^i &= g^i(X_T) + \int_t^T f^i(X_s,(Y_s^n)^i,(Z_s^n)^i)\mathrm{d}s - \int_t^T (Z_s^n)^i.\mathrm{d}W_s \\
&\quad +(K_T^n)^i - (K_t^n)^i , \\
(Y_t^n)^i &\geq (S_t^n)^i , \forall t\in[0,T] \text{ and } \int_0^T ((Y_t^n)^i - (S_t^n)^i)\mathrm{d}(K_t^n)^i = 0 .
\end{aligned}$$

with $S_t^n = \mathcal{P}(X_t, Y_t^{n-1})$, for $n \geq 1$ and $S^0 = -\infty$, meaning that Y^0 is a non-reflected BSDE.

The sequence $(Y^n)_{n\in\mathbb{N}}$ converges increasingly to $\dot{Y}$, see [41] and we have

$$\bar{Y} \succeq Y^n \succeq Y^{n-1} \text{ and } Y_t^n \longrightarrow \dot{Y}_t \quad \text{as } n\to\infty .$$

To obtain (4.4.20), we then prove $Y^d \succeq Y^n$, for all n, using the following induction argument.

(i) Using a comparison argument, it is clear that $Y^d \succeq Y^0$.

Fix $i,j\in\mathcal{I}$ and introduce $\Gamma^0 := (Y^0)^j - c_{ij}$ satisfying

$$\begin{aligned}
\Gamma_t^0 &= (Y_{r_{k+1}}^0)^j - c_{ij} + \int_t^{r_{k+1}} f^j(X_s,\Gamma_s^0 + c_{ij},(Z_s^0)^j)\mathrm{d}s \\
&\quad - \int_t^{r_{k+1}} (Z_s^0)^j.\mathrm{d}W_s , \quad t\in[r_k,r_{k+1}) , \ k\in\{0,\dots,\kappa-1\} .
\end{aligned}$$

Observe that

$$(Y_r^d)^i = \dot{Y}_r^i \geq \dot{Y}_r^j - c_{ij} \geq (Y_r^0)^j - c_{ij} , \quad r\in\Re .$$

By definition $h^i \geq f^j$, using a comparison theorem, we then obtain that $(Y_t^d)^i \geq \Gamma_t^0$ which leads, since i, j are arbitrary, to

$$Y^d \succeq S^1 . \tag{4.4.24}$$

(ii) Assume that $Y^d \succeq S^n$, for some $n > 0$. Therefore Y^d interprets as a BSDE reflected on S^n. Following a comparison argument, we deduce from $Y_T^d \succeq Y_T^n$ and the definitions of h and f that

$$Y^d \ \succeq \ Y^{n+1} . \tag{4.4.25}$$

For $i, j \in \mathcal{I}$, we then introduce $\Gamma^n = (Y^n)^j - c_{ij}$ which satisfies for $k \in \{0, \ldots, \kappa - 1\}$

$$\begin{aligned}
\Gamma^n_t &= (Y^n_{r_{k+1}})^j - c_{ij} + \int_t^{r_{k+1}} f^j(X_s, \Gamma^n_s + c_{ij}, (Z^n_s)^j)\mathrm{d}s \\
&\quad - \int_t^{r_{k+1}} (Z^n_s)^j.\mathrm{d}W_s + (K^n_T)^j - (K^n_t)^j \ , \quad t \in [r_k, r_{k+1}) \ . \\
\Gamma^n_t &\geq (S^n_t)^j \ , \ \forall t \in [0,T] \text{ and } \int_0^T (\Gamma^n_t - (S^n_t)^j)\mathrm{d}(K^n_t)^j = 0 \ .
\end{aligned}$$

Observe that

$$(Y^d_r)^i = \dot{Y}^i_r \geq \dot{Y}^j_r - c_{ij} \geq (Y^n_r)^j - c_{ij} \ , \quad r \in \Re \ .$$

By definition $h^i \geq f^j$, using a comparison theorem, we then obtain that $(Y^d_t)^i \geq \Gamma^n_t$ which leads, since i, j are arbitrary, to

$$Y^d \succeq S^{n+1} \ . \tag{4.4.26}$$

(iii) The proof is then concluded combining (i) and (ii) above. □

4.4.2 Convergence results

Setting $\Re = \pi$, and combining Theorem 4.3.1 with Theorem 4.4.1 and Theorem 4.4.2, we obtain the following results.

Theorem 4.4.3 *If f does not depend on z and $|\pi|L < 1$, the following holds*

$$\sup_{i \leq n} \mathbb{E}\Big[|\dot{Y}_{t_i} - Y^\pi_{t_i}|^2 + |\dot{Y}_{t_i} - \widetilde{Y}^\pi_{t_i}|^2\Big] \ \leq \ C^\varepsilon_L |\pi|^{1-\varepsilon} \ ,$$

for all $\varepsilon > 0$. Moreover, whenever the cost functions are constant, we have

$$\sup_{i \leq n} \mathbb{E}\Big[|\dot{Y}_{t_i} - Y^\pi_{t_i}|^2 + |\dot{Y}_{t_i} - \widetilde{Y}^\pi_{t_i}|^2\Big] \ \leq \ C_L |\pi| \ ,$$

and

$$\sup_{t \in [0,T]} \mathbb{E}\Big[|\dot{Y}_t - Y^\pi_t|^2 + |\dot{Y}_t - \widetilde{Y}^\pi_t|^2\Big] \ \leq \ C_L |\pi|^{\frac{1}{2}} \ .$$

4.5 Appendix: a priori estimates

4.5.1 A priori estimates for continuously and discretely reflected BSDEs

We first prove estimates for BSDEs with reflections in a somehow "abstract" setting. This allows us to retrieve estimate for continuously and discretely reflected BSDEs.

We then consider a process $(Y^g, K^g, Z^g) \in (\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2)^{\mathcal{I}}$ satisfying on $[0,T]$

$$Y^g_t = \xi^g + \int_t^T f^g(s, Y^g_s, Z^g_s)\mathrm{d}s - \int_t^T Z^g_s.\mathrm{d}W_s + K^g_T - K^g_t \ , \tag{4.5.1}$$

where ξ^g is a random variable in $(L^2(\mathcal{F}_T))^{\mathcal{I}}$, the map $(y,z) \mapsto f^g(.,y,z)$ is L-Lipschitz continuous and the random function $s \mapsto f^g(s,0,0)$ belongs to $(\mathcal{H}^{\mathbf{2}})^{\mathcal{I}}$.

Moreover for S^g a continuous process in $\mathcal{S}^2$, (Y^g, K^g) satisfies

$$\int_t^T (Y^g_{u-} - S^g_u).\mathrm{d}K^g_u \quad = \quad 0 \;,\; 0 \le t \le T \;. \tag{4.5.2}$$

Finally, we suppose that the jumps of Y^g and K^g occur only on a finite grid $\Re$ and we work under the following integrability assumption

($\mathbf{H}g$) There exist $p_0 \ge 2$ and a nonnegative random variable β such that

$$|\xi^g| + \sup_{t\in[0,T]} \left(|S^g_t| + |Y^g_t| + \int_t^T |f^g(s,0,0)|\mathrm{d}s \right) \le \beta \;\text{ and }\; \mathbb{E}[|\beta|^{p_0}] \le C_L^{p_0} \;. \tag{4.5.3}$$

Proposition 4.5.1 *In the abstract setting introduced above and under* ($\mathbf{H}g$)*, the following holds*

$$\mathbb{E}\Big[|K^g_T|^{p_0} + (\int_0^T |Z^g_s|^2 \mathrm{d}s)^{\frac{p_0}{2}} \Big] \le C_L^{p_0} \;,$$

recall that C_L^p does not depend on $\Re$.

Proof. Applying Ito's formula to the *càdlàg* process $|Y^g|^2$ on $[0,t]$, we obtain

$$|Y^g_t|^2 = \;\; |Y^g_0|^2 + 2\textstyle\int_{(0,t]} Y^g_{s-}.\mathrm{d}Y^g_s + \int_{(0,t]} |Z^g_s|^2 \mathrm{d}s + \sum_{s\le t}(|Y^g_s|^2 - |Y^g_{s-}|^2 - 2Y^g_{s-}.\Delta Y^g_s) \;,$$

where the last term at the right-hand side is obviously non negative. Recalling (4.5.1), we get

$$\begin{aligned} |Y^g_t|^2 + \int_t^T |Z^g_s|^2 \mathrm{d}s \;\; &\le \;\; |Y^g_T|^2 + 2\int_t^T Y^g_{s-}.f^g(s, Y^g_s, Z^g_s)\mathrm{d}s \\ &\quad +2\int_{(t,T]} Y^g_{s-}.\mathrm{d}K^g_s + 2\int_t^T (Z^g_s Y^g_s).\mathrm{d}W_s. \end{aligned}$$

Using standard arguments, together with (4.5.2) and ($\mathbf{H}g$), we compute that

$$\int_0^T |Z^g_s|^2 \mathrm{d}s \;\; \le \;\; C_L \beta(\beta + K^g_T) + C_L \int_0^T (Z^g_s Y^g_s).\mathrm{d}W_s. \tag{4.5.4}$$

Moreover, we have,

$$|K^g_T|^2 \;\; \le \;\; C_L \left[\beta^2 + \int_0^T |Z^g_s|^2 \mathrm{d}s + (\int_0^T Z^g_s.\mathrm{d}W_s)^2 \right] \tag{4.5.5}$$

Combining (4.5.4) and (4.5.5) we obtain for $\epsilon > 0$,

$$\int_0^T |Z^g_s|^2 \mathrm{d}s \;\; \le \;\; \frac{C_L}{\epsilon}\beta^2 + \epsilon \int_0^T |Z^g_s|^2 \mathrm{d}s + \epsilon \left(\int_0^T Z^g_s.\mathrm{d}W_s \right)^2 + C_L \int_0^T (Z^g_s Y^g_s).\mathrm{d}W_s. \tag{4.5.6}$$

where we used the inequality $ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$ for $a, b \in \mathbb{R}$ and $\epsilon > 0$. We then compute,

$$\begin{aligned}
\mathbb{E}\Big[\big(\int_0^T |Z_s^g|^2 \mathrm{d}s\big)^{\frac{p_0}{2}}\Big] \quad &\le \quad C_L^p\Big(\epsilon^{-\frac{p_0}{2}} + \epsilon^{\frac{p_0}{2}} \mathbb{E}\Big[\big(\int_0^T |Z_s^g|^2 \mathrm{d}s\big)^{\frac{p_0}{2}}\Big] \\
&\quad + \epsilon^{\frac{p_0}{2}} \mathbb{E}\Big[\big(\int_0^T Z_s^g.\mathrm{d}W_s\big)^{p_0}\Big] + C_L \mathbb{E}\Big[\big(\int_0^T (Z_s^g Y_s^g).\mathrm{d}W_s\big)^{\frac{p_0}{2}}\Big]\Big). \quad (4.5.7)
\end{aligned}$$

Using Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathbb{E}\Big[\big(\int_0^T |Z_s^g|^2 \mathrm{d}s\big)^{\frac{p_0}{2}}\Big] \quad &\le C_L^{p_0}\left(\epsilon^{-\frac{p_0}{2}} + \epsilon^{\frac{p_0}{2}} \mathbb{E}\Big[\big(\int_0^T |Z_s^g|^2 \mathrm{d}s\big)^{\frac{p_0}{2}}\Big] + \mathbb{E}\Big[\big(\int_0^T |Z_s^g Y_s^g|^2 \mathrm{d}s\big)^{\frac{p_0}{4}}\Big]\right), \\
&\le C_L^{p_0}\left(\epsilon^{-\frac{p_0}{2}} + \epsilon^{-\frac{p_0}{2}} \mathbb{E}\Big[\sup_{s\in[0,T]} |Y_s^g|^{p_0}\Big] + \epsilon^{\frac{p_0}{2}} \mathbb{E}\Big[\big(\int_0^T |Z_s^g|^2 \mathrm{d}s\big)^{\frac{p_0}{2}}\Big]\right)
\end{aligned}$$

It follows from ($\mathbf{H}g$), for ϵ small enough

$$\mathbb{E}\Big[\big(\int_0^T |Z_s^g|^2 \mathrm{d}s\big)^{\frac{p_0}{2}}\Big] \quad \le \quad C_L^{p_0}. \tag{4.5.8}$$

Taking (4.5.5) up to the power $\frac{p_0}{2}$, and combining Burkholder-Davis-Gundy inequality with the majoration obtained in (4.5.8) yields

$$\mathbb{E}\big[|K_T^g|^{p_0}\big] \quad \le \quad C_L^{p_0},$$

which concludes the proof of the proposition. □

We now apply the previous general result to two more explicit cases of interest.

Proposition 4.5.2 *The solution of the continuously reflected BSDE* (4.4.1) *satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\dot{Y}_t|^p + |\dot{K}_T|^p + \big(\int_0^T |\dot{Z}_s|^2 \mathrm{d}s\big)^{\frac{p}{2}}\Big] \quad \le \quad C_L^p, \qquad p \ge 2\,.$$

Proof.

Step 1. Define the functions $\breve{f} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\breve{g} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\breve{f}^j = \sum_{i=1}^d |f^i| \;\text{ and }\; \breve{g}^j = \sum_{i=1}^d |g^i|\,, \qquad 1 \le j \le d\,,$$

and denote by $(\breve{Y}, \breve{Z}) \in (\mathcal{S}^2 \times \mathcal{H}^2)^{\mathcal{I}}$ the solution to

$$\breve{Y}_t \quad = \quad \breve{g}(X_T) + \int_t^T \breve{f}(X_s, \breve{Y}_s, \breve{Z}_s)\mathrm{d}s - \int_t^T \breve{Z}_s.\mathrm{d}W_s\,, \quad 0 \le t \le T. \tag{4.5.9}$$

Since all the components of $\breve{Y}$ are similar, $\breve{Y} \in \mathcal{C}$. Following the arguments in the proof of Theorem 2.4 in [41], we deduce that $\dot{Y} \preceq \breve{Y}$. Using a comparison argument, we also have $Y^0 \preceq \dot{Y}$ where Y^0 is the solution to the BSDE

$$Y_t^0 \quad = \quad g(X_T) + \int_t^T f(X_s, Y_s^0, Z_s^0)\mathrm{d}s - \int_t^T Z_s^0.\mathrm{d}W_s\,, \quad 0 \le t \le T\,.$$

Using standard arguments on non-reflected BSDEs, we compute that

$$\mathbb{E}\Big[\sup_{s\in[0,T]}|\check{Y}_s|^p\Big]+\mathbb{E}\Big[\sup_{s\in[0,T]}|Y_s^0|^p\Big] \;\leq\; C_L^p\,,\quad p\geq 2\,.$$

and then

$$\mathbb{E}\Bigg[\sup_{s\in[0,T]}|\dot{Y}_s|^p\Bigg] \;\leq\; C_L^p\,,\quad p\geq 2\,. \tag{4.5.10}$$

Step 2. For $i\in\mathcal{I}$, we define $\dot{S}^i:=\max_{j\neq i}(\dot{Y}^j-c_{ij}(X))$ and observe that the component $(\dot{Y}^i,\dot{Z}^i,\dot{K}^i)$ is solution to simply reflected BSDE with lower barrier $\dot{S}^i$, and

$$\dot{Y}^i\geq\dot{S}^i,\;\text{ and }\int_0^T(\dot{Y}_s^i-\dot{S}_s^i)\mathrm{d}\dot{K}_s^i=0\,. \tag{4.5.11}$$

Therefore, the framework of Proposition 4.5.1 is satisfied since (4.5.1), (4.5.2) and $(\mathbf{H}g)$ hold true, and the proof is complete. □

We finally give the proof of Proposition 4.2.1.

Proof of Proposition 4.2.1 Following the same argument as in Step 1 of the proof of Proposition 4.5.2 above, we retrieve

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]}|\widetilde{Y}_t|^p+|\tilde{S}_t|^p\Bigg] \;\leq\; C_L^p,\quad p\geq 2\,, \tag{4.5.12}$$

where $\tilde{S}:=\mathcal{P}(X,\widetilde{Y})$. Observe that $\tilde{S}$ is continuous and

$$\int_t^T(\widetilde{Y}_{u-}-\tilde{S}_u).\mathrm{d}\widetilde{K}_u \;=\; \sum_{r\in\Re,r\geq t}(Y_r-\tilde{S}_r).\Delta\widetilde{K}_r \;=\; 0\,.$$

Therefore (4.5.1), (4.5.2) and $(\mathbf{H}g)$ are satisfied for the process $(\widetilde{Y},Z,\widetilde{K})$ and the proof ends applying Proposition 4.5.1. □

4.5.2 A priori estimates for discrete-time schemes of BSDEs

We state here a uniform bound for the discrete-time scheme of discretely reflected BSDEs We suppose in the sequel that the generator f does not depend on the variable Z.

Proposition 4.5.3 *For $|\pi|L<1$, the following bound holds*

$$\sup_{0\leq i\leq n}\mathbb{E}\left[|\widetilde{Y}_{t_i}^\pi|^p\right] \;\leq\; C_L^p,\quad p\geq 2, \tag{4.5.13}$$

recall that C_L^p neither depends on $\Re$ nor on π.

To prove this proposition, we first need a comparison theorem for discrete-time schemes of BSDEs in the case where the driver does not depend on the variable Z.

For $k = 1, 2$, let ξ_k be a square integrable random variable and $\psi_k : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ a $L-$Lipschitz generator function. We suppose that $\xi_1 \geq \xi_2$ and $\psi_1 \geq \psi_2$ on $\mathbb{R}^m \times \mathbb{R}^d$. For a time grid π, we denote by $(Y^{\pi,k}, \bar{Z}^{\pi,k})$, $k = 1, 2$, the discrete-time scheme:
starting from the terminal condition

$$Y_T^{\pi,k} := \xi_k$$

we compute recursively, for $i \leq n-1$,

$$\begin{cases} \bar{Z}_{t_i}^{\pi,k} &= (t_{i+1}-t_i)^{-1}\mathbb{E}\left[Y_{t_{i+1}}^{\pi,k}(W_{t_{i+1}} - W_{t_i})' \mid \mathcal{F}_{t_i}\right] , \\ Y_{t_i}^{\pi,k} &= \mathbb{E}\left[Y_{t_{i+1}}^{\pi,k} \mid \mathcal{F}_{t_i}\right] + (t_{i+1}-t_i)\psi_k(X_{t_i}^\pi, Y_{t_i}^{\pi,k}) , \end{cases} \tag{4.5.14}$$

Lemma 4.5.1 *For any π such that $|\pi|L < 1$, we have $Y_t^{\pi,1} \geq Y_t^{\pi,2}$ for all $t \in \pi$.*

Proof. Recall that $\pi = \{t_0 = 0, \ldots, t_n = T\}$. Since the results follows from a backward induction on π, we just prove $Y_t^{\pi,1} \geq Y_t^{\pi,2}$. Using (4.5.14), we compute

$$\begin{aligned} Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2} &= \mathbb{E}\left[\xi_1 - \xi_2 \middle| \mathcal{F}_{t_{n-1}}\right] \\ &\quad +(T - t_{n-1})\left[\psi_1(X_{t_{n-1}}^\pi Y_{t_{n-1}}^{\pi,1}) - \psi_2(X_{t_{n-1}}^\pi Y_{t_{n-1}}^{\pi,2})\right] \\ &= \mathbb{E}\left[\xi_1 - \xi_2 \middle| \mathcal{F}_{t_{n-1}}\right] \\ &\quad +(T - t_{n-1})\Lambda_{n-1}\left(Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2}\right) + \Delta_{n-1} , \end{aligned} \tag{4.5.15}$$

where

$$\Lambda_{n-1} = \begin{cases} \frac{\psi_1(X_{t_{n-1}}^\pi Y_{t_{n-1}}^{\pi,1}) - \psi_2(X_{t_{n-1}}^\pi Y_{t_{n-1}}^{\pi,2})}{Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2}} & \text{if} \quad Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2} \neq 0, \\ 0 & \text{else} , \end{cases} \tag{4.5.16}$$

and

$$\Delta_{n-1} = \psi_1(X_{t_{n-1}}^\pi Y_{t_{n-1}}^{\pi,2}) - \psi_2(X_{t_{n-1}}^\pi Y_{t_{n-1}}^{\pi,2}) .$$

Since ψ_k is $L-$Lipschitz, $k = 1, 2$, the condition $|\pi|L < 1$, implies $(T - t_{n-1})\Lambda_{n-1} < 1$. Then using $\xi_1 \geq \xi_2$ and $\psi_1 \geq \psi_2$, we get from (4.5.15) the desired result . □

Proof of Proposition 4.5.3. Using the same arguments as in Step 1. of the proof of Proposition 4.5.2, we obtain, using Lemma 4.5.1, $\breve{Y}^\pi \succeq Y^\pi \succeq Y^{0,\pi}$, where $\breve{Y}^\pi$ and $Y^{0,\pi}$ are the discrete time schemes associated respectively to (4.5.9) and (4.5.10), recall that f does not depend on Z. Thus, to prove (4.5.13), it suffices to prove

$$\sup_{0\leq i\leq n} \mathbb{E}[|\breve{Y}_{t_i}^\pi|^p] + \sup_{0\leq i\leq n} \mathbb{E}[|Y_{t_i}^{0,\pi}|^p] \leq C_L^p, \quad p \geq 2.$$

Since the arguments are the same, we only prove the last bound for the process $Y^{0,\pi}$. Applying Itô's formula to $|Y_s^{0,\pi}|^2$ for $s \in [t_j, t_{j+1}]$ and using standard arguments, we compute for $t \leq t_i$ with $i \leq j$

$$\mathbb{E}[|Y_s^{0,\pi}|^2|\mathcal{F}_t] \quad \leq \quad C_L \mathbb{E}\left[|Y_{t_{j+1}}^{0,\pi}|^2 + \int_s^{t_{j+1}} |Y_u^{0,\pi}|^2 \mathrm{d}u + \int_s^{t_{j+1}} (1 + |Y_{t_j}^{0,\pi}|^2 + |X_{t_j}^\pi|^2) \mathrm{d}u \Big| \mathcal{F}_t\right] .$$

Applying Gronwall's Lemma we get

$$\mathbb{E}[|Y_s^{0,\pi}|^2|\mathcal{F}_t] \quad \leq \quad e^{C_L|\pi|} \mathbb{E}\left[|Y_{t_{j+1}}^{0,\pi}|^2 + \int_s^{t_{j+1}} (1 + |Y_{t_j}^{0,\pi}|^2 + |X_{t_j}^\pi|^2) \mathrm{d}s \Big| \mathcal{F}_t\right],$$

this leads, for $s = t_j$, $t = t_i$ (recall $i \leq j$) and $|\pi|$ small enough, to

$$\mathbb{E}[|Y_{t_j}^{0,\pi}|^2|\mathcal{F}_{t_i}] \quad \leq \quad \frac{e^{C_L|\pi|}}{1 - |\pi|} \mathbb{E}\left[|Y_{t_{j+1}}^{0,\pi}|^2 + |\pi|(1 + \sup_{t \in [0,T]} |X_t^\pi|^2) \Big| \mathcal{F}_{t_i}\right],$$

using an induction argument, we get

$$\mathbb{E}[|Y_{t_j}^{0,\pi}|^2|\mathcal{F}_{t_i}] \quad \leq \quad \frac{e^{C_L|\pi|n}}{1 - |\pi|} \mathbb{E}\left[|g(X_T^\pi)|^2 + n|\pi|(1 + \sup_{t \in [0,T]} |X_t^\pi|^2) \Big| \mathcal{F}_{t_i}\right],$$

By definition $|\pi|n \leq L$, hence we get

$$|Y_{t_j}^{0,\pi}|^2 \quad \leq \quad C_L \left(1 + \mathbb{E}\Big[\sup_{t \in [0,T]} |X_t^\pi|^2 \Big| \mathcal{F}_{t_i}\Big]\right).$$

Taking up the power $\frac{p}{2}$, we get from (4.2.2)

$$\sup_{0 \leq i \leq n} \mathbb{E}\left[|Y^{0,\pi}|^p\right] \quad \leq \quad C_L^p, \quad p \geq 2.$$

□

We now now turn to the proof of Proposition 4.3.2.

Proof of Proposition 4.3.2. Using Proposition 4.2.1 and Proposition 4.5.3, we have

$$\mathbb{E}\left[\int_0^T \left|f(X_{\pi(s)}^\pi, \widetilde{Y}_{\pi(s)}^\pi) \vee f(X_s, \widetilde{Y}_s)\right|^p \mathrm{d}s\right] \quad \leq \quad C_L^p\,, \quad p \geq 2. \tag{4.5.17}$$

We introduce the processes $(\hat{Y}^0, \hat{Z}^0)$ and $(\breve{\hat{Y}}, \breve{\hat{Z}})$ defined respectively by

$$\hat{Y}_t^0 \quad = \quad g(X_T) \vee g(X_T^\pi) + \int_t^T f(X_{\pi(s)}^\pi, \widetilde{Y}_{\pi(s)}^\pi) \vee f(X_s, \widetilde{Y}_s) \mathrm{d}s - \int_t^T \hat{Z}_s^0.\mathrm{d}W_s, \quad 0 \leq t \leq T,$$

and

$$\breve{\hat{Y}}_t \quad = \quad \breve{\hat{g}} + \int_t^T \breve{\hat{f}}_s \mathrm{d}s - \int_t^T \breve{\hat{Z}}_s.\mathrm{d}W_s, \quad 0 \leq t \leq T,$$

where $\breve{\hat{g}} \in (L^p(\mathcal{F}_T))^{\mathcal{I}}$ and $f \in (\mathcal{H}^p)^{\mathcal{I}}$ are defined by are defined by

$$\breve{\hat{g}}^k \;=\; \sum_{i\in\mathcal{I}} |g^i(X_T) \vee g^i(X_T^\pi)| \quad \text{and} \quad \breve{\hat{f}}_s^k \;=\; \sum_{i\in\mathcal{I}} |f^i(X^\pi_{\pi(s)}, \widetilde{Y}^\pi_{\pi(s)}) \vee f^i(X_s, \widetilde{Y}_s)|,$$

for $0 \leq s \leq T$ and $k \in \mathcal{I}$. Following the same argument as in Step 1 of the proof of Proposition 4.5.2 above, we retrieve from (4.5.17), $\breve{\hat{Y}} \succeq \hat{Y} \succeq \hat{Y}^0$ and

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]} |\hat{Y}_t^0|^p\Bigg] + \mathbb{E}\Bigg[\sup_{t\in[0,T]} |\breve{\hat{Y}}_t|^p\Bigg] \;\leq\; C_L^p, \quad p \geq 2\ ,$$

which gives

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]} |\hat{Y}_t|^p\Bigg] \;\leq\; C_L^p, \quad p \geq 2\ . \tag{4.5.18}$$

From (4.5.18) and (4.5.17) we obtain that the discretely reflected BSDE (4.3.10) satisfy $(\mathbf{H}g)$ with $p_0 = p$. The proof is then concluded using Proposition 4.5.1. $\square$

Part III

OPTIMAL PORTFOLIO LIQUIDATION WITH LIQUIDITY RISK

Chapter 5

Optimal portfolio liquidation with execution cost and risk

Abstract : We study the optimal portfolio liquidation problem over a finite horizon in a limit order book with bid-ask spread and temporary market price impact penalizing speedy execution trades. We use a continuous-time modeling framework, but in contrast to previous related papers (see e.g. [72] and [74]), we do not assume continuous-time trading strategies. We consider instead real trading that occur in discrete-time, and this is formulated as an impulse control problem under a solvency constraint, including the lag variable tracking the time interval between trades. A first important result of our paper is to show that nearly optimal execution strategies in this context lead actually to a finite number of trading times, and this holds true without assuming ad hoc any fixed transaction fee. Next, we derive the dynamic programming quasi-variational inequality satisfied by the value function in the sense of constrained viscosity solutions. We also introduce a family of value functions converging to our value function, and which is characterized as the unique constrained viscosity solutions of an approximation of our dynamic programming equation. This convergence result is useful for numerical purpose, postponed in a further study.

Keywords: Optimal portfolio liquidation, execution trade, liquidity effects, order book, impulse control, viscosity solutions.

5.1 Introduction

Understanding trade execution strategies is a key issue for financial market practitioners, and has attracted a growing attention from the academic researchers. An important problem faced by stock traders is how to liquidate large block orders of shares. This is a challenge due to the following dilemma. By trading quickly, the investor is subject to higher costs due to market impact reflecting the depth of the limit order book. Thus, to minimize price impact, it is generally beneficial to break up a large order into smaller blocks. However, more gradual trading over time results in higher risks since the asset value can vary more during the investment horizon in an uncertain environment. There has been recently a considerable interest in the literature on such liquidity effects, taking into account permanent and/or temporary price impact, and problems of this type were studied by Bertsimas and Lo [9], Almgren and Criss [1], Bank and Baum [6], Cetin, Jarrow and Protter [19], Obizhaeva and Wang [58], He and Mamayski [42], Schied an Schöneborn [74], Ly Vath, Mnif and Pham [53], Rogers and Singh [72], and Cetin, Soner and Touzi [20], to mention some of them.

There are essentially two popular formulation types for the optimal trading problem in the literature: discrete-time versus continuous-time. In the discrete-time formulation, we may distinguish papers considering that trading take place at fixed deterministic times (see [9]), at exogenous random discrete times given for example by the jumps of a Poisson process (see [69], [7]), or at discrete times decided optimally by the investor through an impulse control formulation (see [42] and [53]). In this last case, one usually assumes the existence of a fixed transaction cost paid at each trading in order to ensure that strategies do not accumulate in time and occur really at discrete points in time (see e.g. [49] or [59]). The continuous-time trading formulation is not realistic in practice, but is commonly used (as in [19], [74] or [72]), due to the tractability and powerful theory of the stochastic calculus typically illustrated by Itô's formula. In a perfectly liquid market without transaction cost and market impact, continuous-time trading is often justified by arguing that it is a limit approximation of discrete-time trading when the time step goes to zero. However, one may question the validity of such assertion in the presence of liquidity effects.

In this paper, we propose a continuous-time framework taking into account the main liquidity features and risk/cost tradeoff of portfolio execution: there is a bid-ask spread in the limit order book, and temporary market price impact penalizing rapid execution trades. However, in contrast with previous related papers ([74] or [72]), we do not assume continuous-time trading strategies. We consider instead real trading that take place in discrete-time, and without assuming ad hoc any fixed transaction cost, in accordance with the practitioner literature. Moreover, a key issue in line of the banking regulation and solvency constraints is to define in an economically meaningful way the portfolio value of a position in stock at any time, and this is addressed in our modelling. These issues are formulated conveniently through an impulse control problem including the lag variable tracking the time interval between trades. Thus, we combine the advantages of the stochastic calculus techniques, and the realistic modeling of portfolio liquidation. In this context, we study the optimal portfolio liquidation problem over a finite horizon: the investor seeks to unwind an initial position in stock shares by maximizing his expected utility from terminal liquidation wealth, and under a natural economic solvency constraint involving the liquidation value of a portfolio.

A first important result of our paper is to show that nearly optimal execution strategies in this modelling lead actually to a finite number of trading times. While most models dealing with trading strategies via an impulse control formulation assumed fixed transaction cost in order to justify a posteriori the discrete-nature of trading times, we prove here that discrete-time trading appear naturally as a consequence of liquidity features represented by temporary price impact and bid-ask spread. Next, we derive the dynamic programming quasi-variational inequality (QVI) satisfied by the value function in the sense of constrained viscosity solutions in order to handle state constraints. There are some technical difficulties related to the nonlinearity of the impulse transaction function induced by the market price impact, and the non smoothness of the solvency boundary. In particular, since we do not assume a fixed transaction fee, which precludes the existence of a strict supersolution to the QVI, we can not prove directly a comparison principle (hence a uniqueness result) for the QVI. We then consider two types of approximations by introducing families of value functions converging to our original value function, and which are characterized as unique constrained viscosity solutions to their dynamic programming equations. This convergence result is useful for numerical purpose, postponed in a further study.

The plan of the paper is organized as follows. Section 2 presents the details of the model and formulates the liquidation problem. In Section 3, we show some interesting economical and mathematical properties of the model, in particular the finiteness of the number of trading strategies under illiquidity costs. Section 4 is devoted to the dynamic programming and viscosity properties of the value function to our impulse control problem. We propose in Section 5 an approximation of the original problem by considering small fixed transaction fee. Finally, Section 6 describes another approximation of the model with utility penalization by small cost. As a consequence, we obtain that our initial value function is characterized as the minimal constrained viscosity solution to its dynamic programming QVI.

5.2 The model and liquidation problem

We consider a financial market where an investor has to liquidate an initial position of $y > 0$ shares of risky asset (or stock) by time T. He faces with the following risk/cost tradeoff: if he trades rapidly, this results in higher costs for quickly executed orders and market price impact; he can then split the order into several smaller blocks, but is then exposed to the risk of price depreciation during the trading horizon. These liquidity effects received recently a considerable interest starting with the papers by Bertsimas and Lo [9], and Almgren and Criss [1] in a discrete-time framework, and further investigated among others in Obizhaeva and Wang [58], Schied an Schöneborn [74], or Rogers and Singh [72] in a continuous-time model. These papers assume continuous trading with instantaneous trading rate inducing price impact. In a continuous time market framework, we propose here a more realistic modeling by considering that trading takes place at discrete points in time through an impulse control formulation, and with a temporary price impact depending on the time interval between trades, and including a bid-ask spread.

We present the details of the model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions, and supporting a one dimensional Brownian motion W on a finite horizon $[0, T]$, $T < \infty$. We denote by $P = (P_t)$ the market price process of the risky asset, by X_t the amount of money (or cash holdings), by Y_t the number of shares in the stock held by the investor at time t, and by Θ_t the time interval between time t and the last trade before t. We set $\mathbb{R}_+^* = (0, \infty)$ and $\mathbb{R}_-^* = (-\infty, 0)$.

• *Trading strategies.* We assume that the investor can only trade discretely on $[0, T]$. This is modelled through an impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 0}$: $\tau_0 \leq \ldots \leq \tau_n \ldots \leq T$ are nondecreasing stopping times representing the trading times of the investor and ζ_n, $n \geq 0$, are $\mathcal{F}_{\tau_n}$−measurable random variables valued in $\mathbb{R}$ and giving the number of stock purchased if $\zeta_n \geq 0$ or selled if $\zeta_n < 0$ at these times. We denote by $\mathcal{A}$ the set of trading strategies. The sequence (τ_n, ζ_n) may be a priori finite or infinite. Notice also that we do not assume a priori that the sequence of trading times (τ_n) is strictly increasing. We introduce the lag variable tracking the time interval between trades:

$$\Theta_t \;\; = \;\; \inf \{t - \tau_n : \tau_n \leq t\}, \quad t \in [0, T],$$

which evolves according to

$$\Theta_t \; = \; t - \tau_n, \quad \tau_n \leq t < \tau_{n+1}, \qquad \Theta_{\tau_{n+1}} = 0, \quad n \geq 0. \tag{5.2.1}$$

The dynamics of the number of shares invested in stock is given by:

$$Y_t \; = \; Y_{\tau_n}, \qquad \tau_n \leq t < \tau_{n+1}, \qquad Y_{\tau_{n+1}} \; = \; Y_{\tau_{n+1}^-} + \zeta_{n+1}, \quad n \geq 0. \tag{5.2.2}$$

• *Cost of illiquidity.* The market price of the risky asset process follows a geometric Brownian motion:

$$dP_t \;\; = \;\; P_t(bdt + \sigma dW_t), \tag{5.2.3}$$

with constants b and $\sigma > 0$. We do not consider a permanent price impact on the price, i.e. the lasting effect of large trader, but focus here on the effect of illiquidity, that is the price at which an investor will trade the asset. Suppose now that the investor decides at time t

to make an order in stock shares of size e. If the current market price is p, and the time lag from the last order is θ, then the price he actually get for the order e is:

$$Q(e, p, \theta) \quad = \quad p f(e, \theta), \tag{5.2.4}$$

where f is a temporary price impact function from $\mathbb{R} \times [0, T]$ into $\mathbb{R}_+ \cup \{\infty\}$. We assume that the Borelian function f satisfies the following liquidity and transaction cost properties:

(H1f) $f(0, \theta) = 1$, and $f(., \theta)$ is nondecreasing for all $\theta \in [0, T]$,

(H2f) (i) $f(e, 0) = 0$ for $e < 0$, and (ii) $f(e, 0) = \infty$ for $e > 0$,

(H3f) $\kappa_b := \sup_{(e,\theta)\in\mathbb{R}_-^* \times[0,T]} f(e, \theta) < 1$ and $\kappa_a := \inf_{(e,\theta)\in\mathbb{R}_+^* \times[0,T]} f(e, \theta) > 1$.

Condition **(H1f)** means that no trade incurs no impact on the market price, i.e. $Q(0, p, \theta) = p$, and a purchase (resp. a sale) of stock shares induces a cost (resp. gain) greater (resp. smaller) than the market price, which increases (resp. decreases) with the size of the order. In other words, we have $Q(e, p, \theta) \geq$ (resp. $\leq$) p for $e \geq$ (resp. $\leq$) 0, and $Q(., p, \theta)$ is nondecreasing. Condition **(H2f)** expresses the higher costs for immediacy in trading: indeed, the immediate market resiliency is limited, and the faster the investor wants to liquidate (resp. purchase) the asset, the deeper into the limit order book he will have to go, and lower (resp. higher) will be the price for the shares of the asset sold (resp. bought), with a zero (resp. infinite) limiting price for immediate block sale (resp. purchase). Condition **(H2f)** also prevents the investor to pass orders at consecutive immediate times, which is the case in practice. Instead of imposing a fixed arbitrary lag between orders, we shall see that condition **(H2)** implies that trading times are strictly increasing. Condition **(H3f)** captures a transaction cost effect: at time t, P_t is the market or mid-price, $\kappa_b P_t$ is the bid price, $\kappa_a P_t$ is the ask price, and $(\kappa_a - \kappa_b)P_t$ is the bid-ask spead. We also assume some regularity conditions on the temporary price impact function:

(Hcf) (i) f is continuous on $\mathbb{R}^* \times (0, T]$,

(ii) f is C^1 on $\mathbb{R}_-^* \times [0, T]$ and $x \mapsto \dfrac{\partial f}{\partial \theta}$ is bounded on $\mathbb{R}_-^* \times [0, T]$.

A usual form (see e.g. [51], [71], [2]) of temporary price impact and transaction cost function f, suggested by empirical studies is

$$f(e, \theta) \quad = \quad e^{\lambda|\frac{e}{\theta}|^\beta \mathrm{sgn}(e)}\Big(\kappa_a \mathbf{1}_{e>0} + \mathbf{1}_{e=0} + \kappa_b \mathbf{1}_{e<0}\Big), \tag{5.2.5}$$

with the convention $f(0, 0) = 1$. Here $0 < \kappa_b < 1 < \kappa_a$, $\kappa_a - \kappa_b$ is the bid-ask spread parameter, $\lambda > 0$ is the temporary price impact factor, and $\beta > 0$ is the price impact exponent. In our illiquidity modelling, we focus on the cost of trading fast (that is the temporary price impact), and ignore as in Cetin, Jarrow and Protter [19] and Rogers and Singh [72] the permanent price impact of a large trade. This last effect could be included in our model, by assuming a jump of the price process at the trading date, depending on the order size, see e.g. He and Mamayski [42] and Ly Vath, Mnif and Pham [53].

• *Cash holdings.* We assume a zero risk-free return, so that the bank account is constant between two trading times:

$$X_t = X_{\tau_n}, \quad \tau_n \le t < \tau_{n+1}, \;\; n \ge 0. \tag{5.2.6}$$

When a discrete trading $\Delta Y_t = \zeta_{n+1}$ occurs at time $t = \tau_{n+1}$, this results in a variation of the cash amount given by $\Delta X_t := X_t - X_{t^-} = -\Delta Y_t.Q(\Delta Y_t, P_t, \Theta_{t^-})$ due to the illiquidity effects. In other words, we have

$$\begin{aligned} X_{\tau_{n+1}} &= X_{\tau_{n+1}^-} - \zeta_{n+1} Q(\zeta_{n+1}, P_{\tau_{n+1}}, \Theta_{\tau_{n+1}^-}) \\ &= X_{\tau_{n+1}^-} - \zeta_{n+1} P_{\tau_{n+1}} f(\zeta_{n+1}, \tau_{n+1} - \tau_n), \quad n \ge 0. \end{aligned} \tag{5.2.7}$$

Notice that similarly as in the above cited papers dealing with continuous-time trading, we do not assume fixed transaction fees to be paid at each trading. They are practically insignificant with respect to the price impact and bid-ask spread. We can then not exclude a priori trading strategies with immediate trading times, i.e. $\Theta_{\tau_{n+1}^-} = \tau_{n+1} - \tau_n = 0$ for some n. However, notice that under condition **(H2f)**, an immediate sale does not increase the cash holdings, i.e. $X_{\tau_{n+1}} = X_{\tau_{n+1}^-} = X_{\tau_n}$, while an immediate purchase leads to a bankruptcy, i.e. $X_{\tau_{n+1}} = -\infty$.

• *Liquidation value and solvency constraint.* A key issue in portfolio liquidation is to define in an economically meaningful way what is the portfolio value of a position on cash and stocks. In our framework, we impose a no-short sale constraint on the trading strategies, i.e.

$$Y_t \ge 0, \quad 0 \le t \le T,$$

which is in line with the bank regulation following the financial crisis, and we consider the liquidation function $L(x, y, p, \theta)$ representing the net wealth value that an investor with a cash amount x, would obtained by liquidating his stock position $y \ge 0$ by a single block trade, when the market price is p and given the time lag θ from the last trade. It is defined on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0, T]$ by

$$L(x, y, p, \theta) = x + ypf(-y, \theta),$$

and we impose the liquidation constraint on trading strategies:

$$L(X_t, Y_t, P_t, \Theta_t) \ge 0, \quad 0 \le t \le T.$$

We have $L(x, 0, p, \theta) = x$, and under condition **(H2f)**(ii), we notice that $L(x, y, p, 0) = x$ for $y \ge 0$. We naturally introduce the liquidation solvency region:

$$\mathcal{S} = \Big\{ (z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0, T] : \; y > 0 \;\text{ and }\; L(z, \theta) > 0 \Big\}.$$

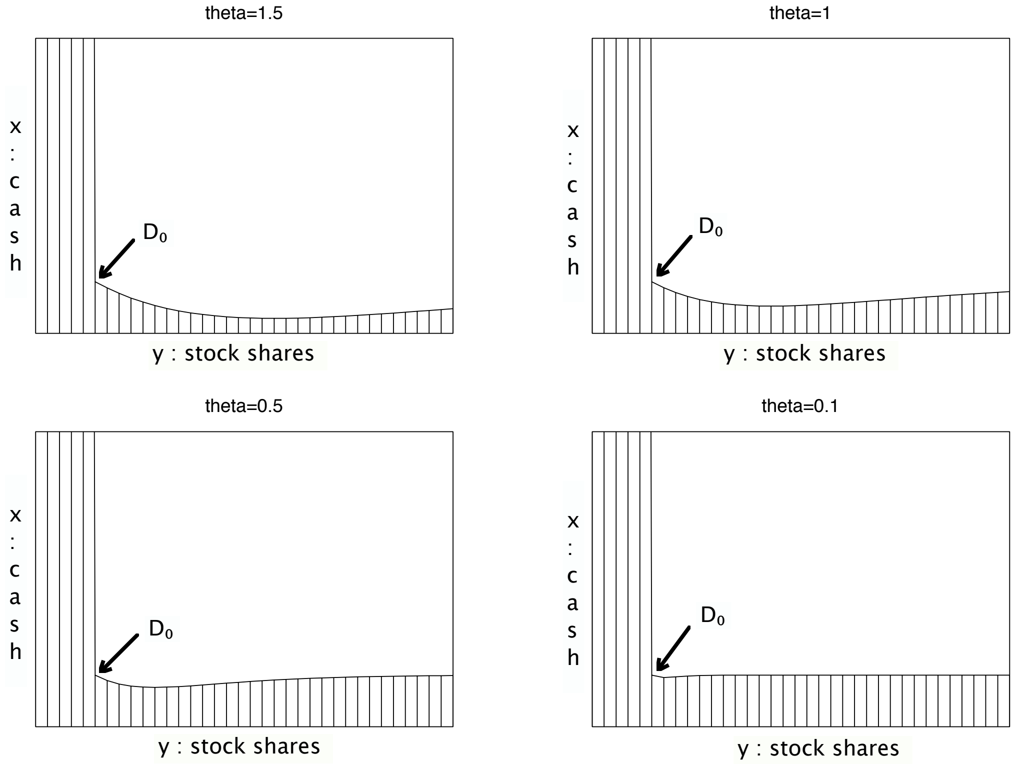

Figure 5.1: Domain $\mathcal{S}$ in the nonhatched zone for fixed $p = 1$ and θ evolving from 1.5 to 0.1. Here $\kappa_b = 0.9$ and $f(e, \theta) = \kappa_b \exp(\frac{e}{\theta})$ for $e < 0$. Notice that when θ goes to 0, the domain converges to the open orthant $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

We denote its boundary and its closure by

$$\partial\mathcal{S} \;=\; \partial_y\mathcal{S} \cup \partial_L\mathcal{S} \quad \text{and} \quad \bar{\mathcal{S}} \;=\; \mathcal{S} \cup \partial\mathcal{S},$$

where

$$\begin{aligned}
\partial_y\mathcal{S} &= \Big\{(z,\theta) = (x,y,p,\theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0,T] :\; y = 0 \;\text{ and }\; x = L(z,\theta) \geq 0\Big\}, \\
\partial_L\mathcal{S} &= \Big\{(z,\theta) = (x,y,p,\theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0,T] :\; L(z,\theta) = 0\Big\}.
\end{aligned}$$

We also denote by D_0 the corner line in $\partial\mathcal{S}$:

$$D_0 \;=\; \{0\} \times \{0\} \times \mathbb{R}_+^* \times [0,T] \;=\; \partial_y\mathcal{S} \cap \partial_L\mathcal{S}.$$

• *Admissible trading strategies.* Given $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}$, we say that the impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n\geq 0}$ is admissible, denoted by $\alpha \in \mathcal{A}(t,z,\theta)$, if $\tau_0 = t - \theta$, τ_n

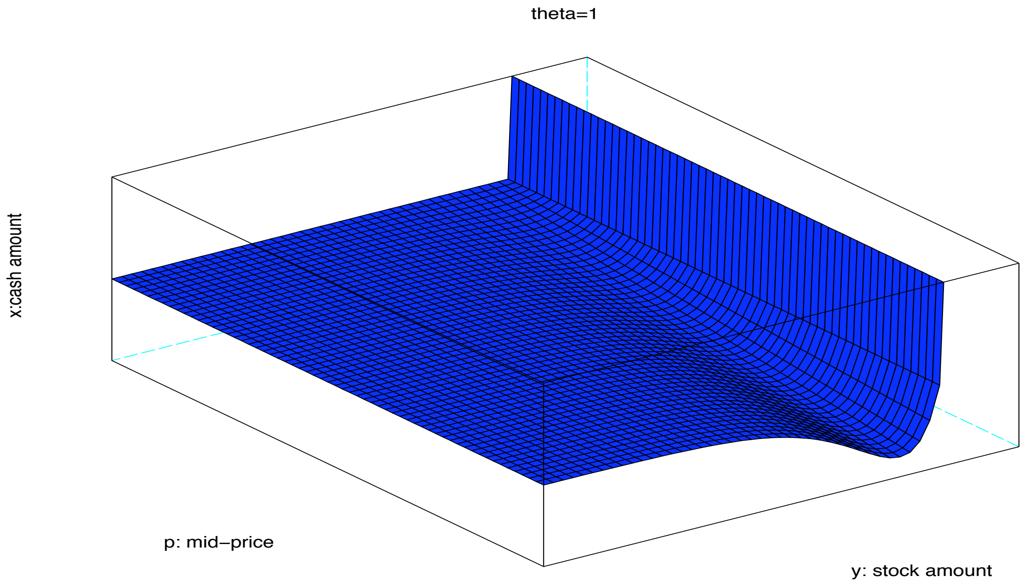

Figure 5.2: Lower bound of the domain $\mathcal{S}$ for fixed $\theta = 1$. Here $\kappa_b = 0.9$ and $f(e, \theta) = \kappa_b \exp(\frac{e}{\theta})$ for $e < 0$. Notice that when p is fixed, we obtain the Figure 1.

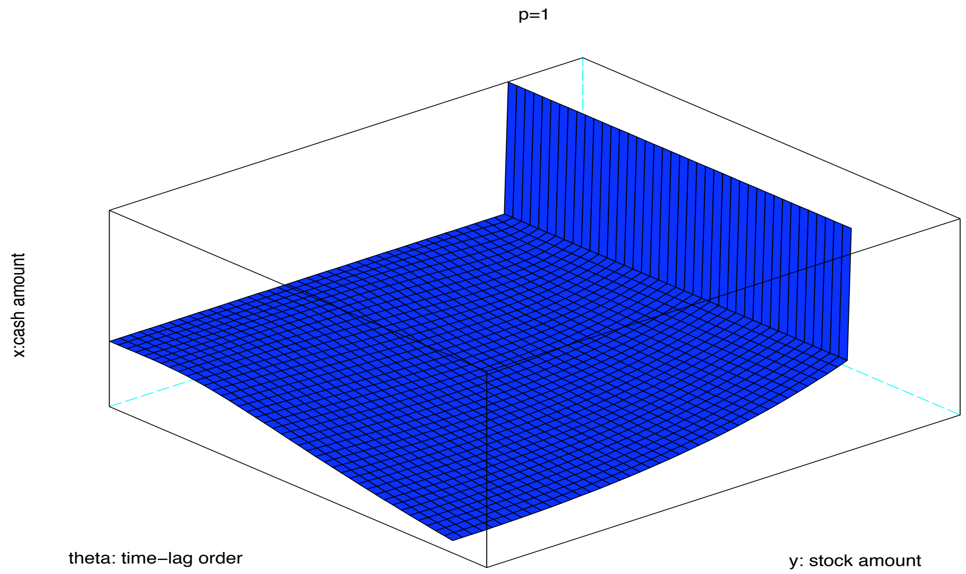

Figure 5.3: Lower bound of the domain $\mathcal{S}$ for fixed $p = 1$ with $f(e, \theta) = \kappa_b \exp(\frac{e}{\theta})$ for $e < 0$ and $\kappa_b = 0.9$. Notice that when θ is fixed, we obtain the Figure 1.

$\geq t$, $n \geq 1$, and the process $\{(Z_s, \Theta_s) = (X_s, Y_s, P_s, \Theta_s), t \leq s \leq T\}$ solution to (5.2.1)-(5.2.2)-(5.2.3)-(5.2.6)-(5.2.7), with an initial state $(Z_{t^-}, \Theta_{t^-}) = (z, \theta)$ (and the convention that $(Z_t, \Theta_t) = (z, \theta)$ if $\tau_1 > t$), satisfies $(Z_s, \Theta_s) \in [0,T] \times \bar{\mathcal{S}}$ for all $s \in [t,T]$. As usual, to alleviate notations, we omitted the dependence of (Z, Θ) in (t, z, θ, α), when there is no ambiguity.

Remark 5.2.1 Let $(t, z, \theta) \in [0,T] \times \bar{\mathcal{S}}$, and consider the impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 0}$, $\tau_0 = t - \theta$, consisting in liquidating immediately all the stock shares, and then doing no transaction anymore, i.e. $(\tau_1, \zeta_1) = (t, -y)$, and $\zeta_n = 0$, $n \geq 2$. The associated state process $(Z = (X, Y, P), \Theta)$ satisfies $X_s = L(z, \theta)$, $Y_s = 0$, which shows that $L(Z_s, \Theta_s)$ $= X_s = L(z, \theta) \geq 0$, $t \leq s \leq T$, and thus $\alpha \in \mathcal{A}(t, z, \theta) \neq \emptyset$.

• *Portfolio liquidation problem.* We consider a utility function U from $\mathbb{R}_+$ into $\mathbb{R}$, nondecreasing, concave, with $U(0) = 0$, and s.t. there exists $K \geq 0$ and $\gamma \in [0, 1)$:

$$\textbf{(HU)} \qquad 0 \ \leq \ U(x) \ \leq \ Kx^\gamma, \quad \forall x \in \mathbb{R}_+.$$

The problem of optimal portfolio liquidation is formulated as

$$v(t, z, \theta) \ = \ \sup_{\alpha \in \mathcal{A}_\ell(t,z,\theta)} \mathbb{E}[U(X_T)], \quad (t, z, \theta) \in [0,T] \times \bar{\mathcal{S}}, \tag{5.2.8}$$

where $\mathcal{A}_\ell(t, z, \theta) = \{\alpha \in \mathcal{A}(t, z, \theta) : \ Y_T \ = \ 0\}$ is nonempty by Remark 5.2.1. Notice that for $\alpha \in \mathcal{A}_\ell(t, z, \theta)$, $X_T = L(Z_T, \Theta_T) \geq 0$, so that the expectations in (5.2.8), and the value function v are well-defined in $[0, \infty]$. Moreover, by considering the particular strategy described in Remark 5.2.1, which leads to a final liquidation value $X_T = L(z, \theta)$, we obtain a lower-bound for the value function;

$$v(t, z, \theta) \ \geq \ U(L(z, \theta)), \quad (t, z, \theta) \in [0,T] \times \bar{\mathcal{S}}. \tag{5.2.9}$$

Remark 5.2.2 We can shift the terminal liquidation constraint in $\mathcal{A}_\ell(t, z, \theta)$ to a terminal liquidation utility by considering the function U_L defined on $\bar{\mathcal{S}}$ by:

$$U_L(z, \theta) \ = \ U(L(z, \theta)), \quad (z, \theta) \in \bar{\mathcal{S}}.$$

Then, problem (5.2.8) is written equivalently in

$$\bar{v}(t, z, \theta) \ = \ \sup_{\alpha \in \mathcal{A}(t,z,\theta)} \mathbb{E}\Big[U_L(Z_T, \Theta_T)\Big], \quad (t, z, \theta) \in [0,T] \times \bar{\mathcal{S}}. \tag{5.2.10}$$

Indeed, by observing that for all $\alpha \in \mathcal{A}_\ell(t, z, \theta)$, we have $\mathbb{E}[U(X_T)] = \mathbb{E}[U_L(Z_T, \Theta_T)]$, and since $\mathcal{A}_\ell(t, z, \theta) \subset \mathcal{A}(t, z, \theta)$, it is clear that $v \leq \bar{v}$. Conversely, for any $\alpha \in \mathcal{A}(t, z, \theta)$ associated to the state controlled process (Z, Θ), consider the impulse control strategy $\tilde{\alpha} = \alpha \cup (T, -Y_T)$ consisting in liquidating all the stock shares Y_T at time T. The corresponding state process $(\tilde{Z}, \tilde{\Theta})$ satisfies clearly: $(\tilde{Z}_s, \tilde{\Theta}_s) = (Z_s, \Theta_s)$ for $t \leq s < T$, and $\tilde{X}_T = L(Z_T, \Theta_T)$, $\tilde{Y}_T = 0$, and so $\tilde{\alpha} \in \mathcal{A}_\ell(t, z, \theta)$. We deduce that $\mathbb{E}[U_L(Z_T, \Theta_T)] = \mathbb{E}[U(\tilde{X}_T)]$ $\leq v(t, z, \theta)$, and so by arbitrariness of α in $\mathcal{A}(t, z, \theta)$, $\bar{v}(t, z, \theta) \leq v(t, z, \theta)$. This proves the equality $v = \bar{v}$. Actually, the above arguments also show that $\sup_{\alpha \in \mathcal{A}_\ell(t,z,\theta)} U(X_T) = \sup_{\alpha \in \mathcal{A}(t,z,\theta)} U_L(Z_T, \Theta_T)$.

Remark 5.2.3 A continuous-time trading version of our illiquid market model with stock price P and temporary price impact f can be formulated as follows. The trading strategy is given by a $\mathbb{F}$-adated process $\eta = (\eta_t)_{0\leq t\leq T}$ representing the instantaneous trading rate, which means that the dynamics of the cumulated number of stock shares Y is governed by:

$$dY_t \quad = \quad \eta_t dt. \tag{5.2.11}$$

The cash holdings X follows

$$dX_t \quad = \quad -\eta_t P_t f(\eta_t) dt. \tag{5.2.12}$$

Notice that in a continuous-time trading formulation, the time interval between trades is $\Theta_t = 0$ at any time t. Under condition **(H2f)**, the liquidation value is then given at any time t by:

$$L(X_t, Y_t, P_t, 0) \quad = \quad X_t, \quad 0 \leq t \leq T,$$

and does not capture the position in stock shares, which is economically not relevant. On the contrary, by explicitly considering the time interval between trades in our discrete-time trading formulation, we take into account the position in stock.

5.3 Properties of the model

In this section, we show that the illiquid market model presented in the previous section displays some interesting and economically meaningful properties on the admissible trading strategies and the optimal performance, i.e. the value function. Let us consider the impulse transaction function Γ defined on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0,T] \times \mathbb{R}$ into $\mathbb{R} \cup \{-\infty\} \times \mathbb{R} \times \mathbb{R}_+^*$ by:

$$\Gamma(z, \theta, e) \quad = \quad \Big(x - epf(e, \theta), y + e, p\Big),$$

for $z = (x, y, p)$, and set $\bar{\Gamma}(z, \theta, e) = (\Gamma(z, \theta, e), 0)$. This corresponds to the value of the state variable (Z, Θ) immediately after a trading at time $t = \tau_{n+1}$ of ζ_{n+1} shares of stock, i.e. $(Z_{\tau_{n+1}}, \Theta_{\tau_{n+1}}) = (\Gamma(Z_{\tau_{n+1}^-}, \Theta_{\tau_{n+1}^-}, \zeta_{n+1}), 0)$. We then define the set of admissible transactions:

$$\mathcal{C}(z, \theta) \quad = \quad \Big\{e \in \mathbb{R} : (\Gamma(z, \theta, e), 0) \in \bar{\mathcal{S}}\Big\}, \qquad (z, \theta) \in \bar{\mathcal{S}}.$$

This means that for any $\alpha = (\tau_n, \zeta_n)_{n\geq 0} \in \mathcal{A}(t, z, \theta)$ with associated state process (Z, Θ), we have $\zeta_n \in \mathcal{C}(Z_{\tau_n^-}, \Theta_{\tau_n^-})$, $n \geq 1$. We define the impulse operator $\mathcal{H}$ by

$$\mathcal{H}\varphi(t, z, \theta) \quad = \quad \sup_{e\in\mathcal{C}(z,\theta)} \varphi(t, \Gamma(z, \theta, e), 0), \qquad (t, z, \theta) \in [0, T] \times \bar{\mathcal{S}}.$$

We also introduce the liquidation function of the (perfectly liquid) Merton model:

$$L_M(z) \quad = \quad x + py, \qquad \forall z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*.$$

For $(t, z = (x, y, p), \theta) \in [0, T] \times \bar{\mathcal{S}}$, we denote by $(Z^{0,t,z}, \Theta^{0,t,\theta})$ the state process starting from (z, θ) at time t, and without any impulse control strategy: it is given by

$$\left(Z_s^{0,t,z}, \Theta_s^{0,t,\theta}\right) = (x, y, P_s^{t,p}, \theta + s - t), \quad t \le s \le T,$$

where $P^{t,p}$ is the solution to (5.2.3) starting from p at time t. Notice that $(Z^{0,t,z}, \Theta^{0,t,\theta})$ is the continuous part of the state process (Z, Θ) controlled by $\alpha \in \mathcal{A}(t, z, \theta)$. The infinitesimal generator $\mathcal{L}$ associated to the process $(Z^{0,t,z}, \Theta^{0,t,\theta})$ is

$$\mathcal{L}\varphi + \frac{\partial \varphi}{\partial \theta} = bp\frac{\partial \varphi}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 \varphi}{\partial p^2} + \frac{\partial \varphi}{\partial \theta}.$$

We first prove a useful result on the set of admissible transactions.

Lemma 5.3.1 *Assume that* **(H1f)**, **(H2f)** *and* **(H3f)** *hold. Then, for all* $(z = (x, y, p), \theta) \in \bar{\mathcal{S}}$, *the set* $\mathcal{C}(z, \theta)$ *is compact in* $\mathbb{R}$ *and satisfy*

$$\mathcal{C}(z, \theta) \subset [-y, \bar{e}(z, \theta)], \tag{5.3.1}$$

where $-y \le \bar{e}(z, \theta) < \infty$ *is given by*

$$\bar{e}(z, \theta) = \begin{cases} \sup\left\{e \in \mathbb{R} : \ epf(e, \theta) \le x\right\}, & \text{if } \theta > 0 \\ 0, & \text{if } \theta = 0. \end{cases}$$

For $\theta = 0$, (5.3.1) *becomes an equality :* $\mathcal{C}(z, 0) = [-y, 0]$.

The set function $\mathcal{C}$ *is continous for the Hausdorff metric, i.e. if* (z_n, θ_n) *converges to* (z, θ) *in* $\bar{\mathcal{S}}$, *and* (e_n) *is a sequence in* $\mathcal{C}(z_n, \theta_n)$ *converging to* e, *then* $e \in \mathcal{C}(z, \theta)$. *Moreover, if* $e \in \mathbb{R} \mapsto ef(e, \theta)$ *is strictly increasing for* $\theta \in (0, T]$, *then for* $(z = (x, y, p), \theta) \in \partial_L \mathcal{S}$ *with* $\theta > 0$, *we have* $\bar{e}(z, \theta) = -y$, *i.e.* $\mathcal{C}(z, \theta) = \{-y\}$.

Proof. By definition of the impulse transaction function Γ and the liquidation function L, we immediately see that the set of admissible transactions is written as

$$\begin{aligned} \mathcal{C}(z, \theta) &= \left\{e \in \mathbb{R} : \ x - epf(e, \theta) \ge 0, \ \text{ and } \ y + e \ge 0\right\} \\ &= \left\{e \in \mathbb{R} : \ epf(e, \theta) \le x\right\} \cap [-y, \infty) =: \mathcal{C}_1(z, \theta) \cap [-y, \infty). \end{aligned} \tag{5.3.2}$$

It is clear that $\mathcal{C}(z, \theta)$ is closed and bounded, thus a compact set. Under **(H1f)** and **(H3f)**, we have $\lim_{e \to \infty} epf(e, \theta) = \infty$. Hence we get $\bar{e}(z, \theta) < \infty$ and $\mathcal{C}_1(z, \theta) \subset (-\infty, \bar{e}(z, \theta)]$. From (5.3.2), we get (5.3.1). Suppose $\theta = 0$. Under **(H2f)**, using $(z, \theta) \in \bar{\mathcal{S}}$, we have $\mathcal{C}_1(z, \theta) = \mathbb{R}_-$. From (5.3.2), we get $\mathcal{C}(z, \theta) = [-y, 0]$.

Let us now prove the continuity of the set of admissible transactions. Consider a sequence $(z_n = (x_n, y_n, p_n), \theta_n)$ in $\bar{\mathcal{S}}$ converging to $(z, \theta) \in \bar{\mathcal{S}}$, and a sequence (e_n) in $\mathcal{C}(z_n, \theta_n)$ converging to e. Suppose first that $\theta > 0$. Then, for n large enough, $\theta_n > 0$ and by observing that $(z, \theta, e) \mapsto \bar{\Gamma}(z, \theta, e)$ is continuous on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}$, we immediately deduce that $e \in \mathcal{C}(z,\theta)$. In the case $\theta = 0$, writing $x_n - e_n f(e_n,\theta_n) \geq 0$, using **(H2f)**(ii) and sending n to infinity, we see that e should necessarily be nonpositive. By writing also that $y_n + e_n \geq 0$, we get by sending n to infinity that $y + e \geq 0$, and therefore $e \in \mathcal{C}(z,0)$ $= [-y, 0]$.

Suppose finally that $e \in \mathbb{R} \mapsto ef(e,\theta)$ is increasing, and fix $(z = (x,y,p),\theta) \in \partial_L \mathcal{S}$, with $\theta > 0$. Then, $L(z,\theta) = 0$, i.e. $x = -ypf(-y,\theta)$. Set $\bar{e} = \bar{e}(z,\theta)$. By writing that $\bar{e}pf(\bar{e},\theta) \leq x = -ypf(-y,\theta)$, and $\bar{e} \geq -y$, we deduce from the increasing monotonicity of $e \mapsto epf(e,\theta)$ that $\bar{e} = -y$. □

Remark 5.3.1 The previous Lemma implies in particular that $\mathcal{C}(z,0) \subset \mathbb{R}_-$, which means that an admissible transaction after an immediate trading should be necessarily a sale. In other words, given $\alpha = (\tau_n, \zeta_n)_{n\geq 0} \in \mathcal{A}(t,z,\theta)$, $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}$, if $\Theta_{\tau_n^-} = 0$, then ζ_n ≤ 0. The continuity property of $\mathcal{C}$ ensures that the operator $\mathcal{H}$ preserves the lower and upper-semicontinuity (see Appendix). This Lemma also asserts that, under the assumption of increasing monotonicity of $e \to ef(e,\theta)$, when the state is in the boundary $L = 0$, then the only admissible transaction is to liquidate all stock shares. This increasing monotonicity means that the amount traded is increasing with the size of the order. Such an assumption is satisfied in the example (5.2.5) of temporary price impact function f for $\beta = 2$, but is not fulfilled for $\beta = 1$. In this case, the presence of illiquidity cost implies that it may be more advantageous to split the order size.

We next state some useful bounds on the liquidation value associated to an admissible transaction.

Lemma 5.3.2 *Assume that* **(H1f)** *holds. Then, we have for all* $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}$*:*

$$0 \;\leq\; L(z,\theta) \;\leq\; L_M(z), \tag{5.3.3}$$

$$L_M(\Gamma(z,\theta,e)) \;\leq\; L_M(z), \quad \forall e \in \mathbb{R}, \tag{5.3.4}$$

$$\sup_{\alpha \in \mathcal{A}(t,z,\theta)} L(Z_s,\Theta_s) \;\leq\; L_M(Z_s^{0,t,z}), \quad t \leq s \leq T. \tag{5.3.5}$$

Furthermore, under **(H3f)***, we have for all* $(z = (x,y,p),\theta) \in \bar{\mathcal{S}}$*,*

$$L_M(\Gamma(z,\theta,e)) \;\leq\; L_M(z) - \min(\kappa_a - 1, 1 - \kappa_b)|e|p, \quad \forall e \in \mathbb{R}. \tag{5.3.6}$$

Proof. Under **(H1f)**, we have $f(e,\theta) \leq 1$ for all $e \leq 0$, which shows clearly (5.3.3). From the definition of L_M and Γ, we see that for all $e \in \mathbb{R}$,

$$L_M(\Gamma(z,\theta,e)) - L_M(z) \;=\; ep\Big(1 - f(e,\theta)\Big), \tag{5.3.7}$$

which yields the inequality (5.3.4). Fix some arbitrary $\alpha = (\tau_n,\zeta_n)_{n\geq 0} \in \mathcal{A}(t,z,\theta)$ associated to the controlled state process (Z,Θ). When a transaction occurs at time $s = \tau_n$, $n \geq 1$, the jump of $L_M(Z)$ is nonpositive by (5.3.4):

$$\Delta L_M(Z_s) \;=\; L_M(Z_{\tau_n}) - L_M(Z_{\tau_n^-}) \;=\; L_M(\Gamma(Z_{\tau_n^-},\Theta_{\tau_n^-},\zeta_n)) - L_M(Z_{\tau_n^-}) \;\leq\; 0.$$

We deduce that the process $L_M(Z)$ is smaller than its continuous part equal to $L_M(Z^{0,t,z})$, and we then get (5.3.5) with (5.3.3). Finally, under the additional condition **(H3f)**, we easily obtain inequality (5.3.6) from relation (5.3.7). □

We now check that our liquidation problem is well-posed by stating a natural upper-bound on the optimal performance, namely that the value function in our illiquid market model is bounded by the usual Merton bound in a perfectly liquid market.

Proposition 5.3.1 *Assume that* **(H1f)** *and* **(HU)** *hold. Then, for all* $(t,z,\theta) \in [0,T]\times\bar{\mathcal{S}}$, *the family* $\{U_L(Z_T,\Theta_T), \alpha \in \mathcal{A}(t,z,\theta)\}$ *is uniformly integrable, and we have*

$$\begin{aligned} v(t,z,\theta) \;\leq\; v_0(t,z) &\;:=\; \mathbb{E}\Big[U\Big(L_M(Z_T^{0,t,z})\Big)\Big], \quad (t,z,\theta) \in [0,T]\times\bar{\mathcal{S}}, \\ &\;\leq\; Ke^{\rho(T-t)}L_M(z)^\gamma, \end{aligned} \tag{5.3.8}$$

where ρ *is a positive constant s.t.*

$$\rho \;\geq\; \frac{\gamma}{1-\gamma}\frac{b^2}{2\sigma^2}. \tag{5.3.9}$$

Proof. From (5.3.5) and the nondecreasing monotonicity of U, we have for all $(t,z,\theta) \in [0,T]\times\bar{\mathcal{S}}$:

$$\sup_{\alpha\in\mathcal{A}_\ell(t,z,\theta)} U(X_T) \;=\; \sup_{\alpha\in\mathcal{A}(t,z,\theta)} U_L(Z_T,\Theta_T) \;\leq\; U(L_M(Z_T^{0,t,z})),$$

and all the assertions of the Proposition will follow once we prove the inequality (5.3.8). For this, consider the nonnegative function φ defined on $[0,T]\times\bar{\mathcal{S}}$ by:

$$\varphi(t,z,\theta) \;=\; e^{\rho(T-t)}L_M(z)^\gamma \;=\; e^{\rho(T-t)}(x+py)^\gamma,$$

and notice that φ is smooth C^2 on $[0,T]\times(\bar{\mathcal{S}}\setminus D_0)$. We claim that for $\rho > 0$ large enough, the function φ satisfies:

$$-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial\theta} - \mathcal{L}\varphi \;\geq\; 0, \qquad \text{on } [0,T]\times(\bar{\mathcal{S}}\setminus D_0).$$

Indeed, a straightforward calculation shows that for all $(t,z,\theta) \in [0,T]\times(\bar{\mathcal{S}}\setminus D_0)$:

$$\begin{aligned} &-\frac{\partial\varphi}{\partial t}(t,z,\theta) - \frac{\partial\varphi}{\partial\theta}(t,z,\theta) - \mathcal{L}\varphi(t,z,\theta) \\ &= e^{\rho(T-t)}L_M(z)^{\gamma-2}\Big[\Big(\sqrt{\rho}L_M(z) + \frac{b\gamma}{2\sqrt{\rho}}yp\Big)^2 + \Big(\frac{\gamma(1-\gamma)\sigma^2}{2} - \frac{b^2\gamma^2}{4\rho}\Big)y^2p^2\Big] \end{aligned} \tag{5.3.10}$$

which is nonegative under condition (5.3.9).

Fix some $(t,z,\theta) \in [0,T]\times\bar{\mathcal{S}}$. If $(z,\theta) = (0,0,p,\theta) \in D_0$, then we clearly have $v_0(t,z,\theta)$ $= U(0)$, and inequality (5.3.8) is trivial. Otherwise, if $(z,\theta) \in \bar{\mathcal{S}}\setminus D_0$, then the process $(Z^{0,t,z},\Theta^{0,t,\theta})$ satisfy $L_M(Z^{0,t,z},\Theta^{0,t,\theta}) > 0$. Indeed, Denote by $(\bar{Z}^{t,z},\bar{\Theta}^{t,\theta})$ the process starting from (z,θ) at t and associated to the strategy consisting in liquidating all stock shares at t. Then we have $(\bar{Z}_s^{t,z}, \bar{\Theta}_s^{t,\theta}) \in \bar{\mathcal{S}} \setminus D_0$ for all $s \in [t,T]$ and hence $L_M(\bar{Z}_s^{t,z}, \bar{\Theta}_s^{t,\theta}) > 0$ for all $s \in [t,T]$. Using (5.3.5) we get $L_M(Z_s^{0,t,z}, \Theta_s^{0,t,\theta}) \geq L_M(\bar{Z}_s^{t,z}, \bar{\Theta}_s^{t,\theta}) > 0$.

We can then apply Itô's formula to $\varphi(s, Z_s^{0,t,z}, \Theta_s^{0,t,\theta})$ between t and $T_R = \inf\{s \geq t : |Z_s^{0,t,z}| \geq R\} \wedge T$:

$$\begin{aligned} \mathbb{E}[\varphi(T_R, Z_{T_R}^{0,t,z}, \Theta_{T_R}^{0,t,\theta})] &= \varphi(t,z) + \mathbb{E}\Big[\int_t^{T_R} \Big(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial \theta} + \mathcal{L}\varphi\Big)(s, Z_s^{0,t,z}, \Theta_s^{0,t,\theta}) ds\Big] \\ &\leq \varphi(t,z). \end{aligned}$$

(The stochastic integral term vanishes in expectation since the integrand is bounded before T_R). By sending R to infinity, we get by Fatou's lemma and since $\varphi(T,z,\theta) = L_M(z)^\gamma$:

$$\mathbb{E}\Big[L_M(Z_T^{0,t,z})^\gamma\Big] \quad \leq \quad \varphi(t,z,\theta).$$

We conclude with the growth condition **(HU)**. □

As a direct consequence of the previous proposition, we obtain the continuity of the value function on the boundary $\partial_y \mathcal{S}$, i.e. when we start with no stock shares.

Corollary 5.3.1 *Assume that* **(H1f)** *and* **(HU)** *hold. Then, the value function v is continuous on $[0,T] \times \partial_y \mathcal{S}$, and we have*

$$v(t,z,\theta) \quad = \quad U(x), \quad \forall t \in [0,T], (z,\theta) = (x,0,p,\theta) \in \partial_y \mathcal{S}.$$

In particular, we have $v(t,z,\theta) = U(0) = 0$, for all $(t,z,\theta) \in [0,T] \times D_0$.

Proof. From the lower-bound (5.2.9) and the upper-bound in Proposition 5.3.1, we have for all $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}$,

$$U\Big(x + ypf(-y,\theta)\Big) \;\leq\; v(t,z,\theta) \;\leq\; \mathbb{E}[U(L_M(Z_T^{0,t,z}))] \;=\; \mathbb{E}[U(x + yP_T^{t,p})].$$

These two inequalities imply the required result. □

The following result states the finiteness of the total number of shares and amount traded.

Proposition 5.3.2 *Assume that* **(H1f)** *and* **(H3f)** *hold. Then, for any $\alpha = (\tau_n, \zeta_n)_{n \geq 0} \in \mathcal{A}(t,z,\theta)$, $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}$, we have*

$$\sum_{n \geq 1} |\zeta_n| \;<\; \infty, \quad \sum_{n \geq 1} |\zeta_n| P_{\tau_n} \;<\; \infty, \quad \textit{and} \quad \sum_{n \geq 1} |\zeta_n| P_{\tau_n} f\Big(\zeta_n, \Theta_{\tau_n^-}\Big) \;<\; \infty, \quad \textit{a.s.}$$

Proof. Fix $(t, z = (x,y,p), \theta) \in [0,T] \times \bar{\mathcal{S}}$, and $\alpha = (\tau_n, \zeta_n)_{n \geq 0} \in \mathcal{A}(t,z,\theta)$. Observe first that the continuous part of the process $L_M(Z)$ is $L_M(Z^{0,t,z})$, and we denote its jump at time τ_n by $\Delta L_M(Z_{\tau_n}) = L_M(Z_{\tau_n}) - L_M(Z_{\tau_n^-})$. From the estimates (5.3.3) and (5.3.6) in Lemma 5.3.2, we then have almost surely for all $n \geq 1$,

$$\begin{aligned} 0 \ \leq \ L_M(Z_{\tau_n}) &= L_M(Z^{0,t,z}_{\tau_n}) + \sum_{k=1}^n \Delta L_M(Z_{\tau_k}) \\ &\leq L_M(Z^{0,t,z}_{\tau_n}) - \bar{\kappa} \sum_{k=1}^n |\zeta_k| P_{\tau_k}, \end{aligned}$$

where we set $\bar{\kappa} = \min(\kappa_a - 1, 1 - \kappa_b) > 0$. We deduce that for all $n \geq 1$,

$$\sum_{k=1}^n |\zeta_k| P_{\tau_k} \ \leq \ \frac{1}{\bar{\kappa}} \sup_{s \in [t,T]} L_M(Z^{0,t,z}_s) \ = \ \frac{1}{\bar{\kappa}} (x + y \sup_{s \in [t,T]} P^{t,p}_s) \ < \ \infty, \quad a.s.$$

This shows the almost sure convergence of the series $\sum_n |\zeta_n| P_{\tau_n}$. Moreover, since the price process P is continous and strictly positive, we also obtain the convergence of the series $\sum_n |\zeta_n|$. Recalling that $f(e, \theta) \leq 1$ for all $e \leq 0$ and $\theta \in [0, T]$, we have for all $n \geq 1$.

$$\begin{aligned} \sum_{k=1}^n |\zeta_k| P_{\tau_k} f(\zeta_k, \Theta_{\tau_k^-}) &= \sum_{k=1}^n \zeta_k P_{\tau_k} f(\zeta_k, \Theta_{\tau_k^-}) + 2 \sum_{k=1}^n |\zeta_k| P_{\tau_k} f(\zeta_k, \Theta_{\tau_k^-}) \mathbf{1}_{\zeta_k \leq 0} \\ &\leq \sum_{k=1}^n \zeta_k P_{\tau_k} f(\zeta_k, \Theta_{\tau_k^-}) + 2 \sum_{k=1}^n |\zeta_k| P_{\tau_k}. \end{aligned} \tag{5.3.11}$$

On the other hand, we have

$$\begin{aligned} 0 \ \leq \ L_M(Z_{\tau_n}) &= X_{\tau_n} + Y_{\tau_n} P_{\tau_n} \\ &= x - \sum_{k=1}^n \zeta_k P_{\tau_k} f(\zeta_k, \Theta_{\tau_k^-}) + (y + \sum_{k=1}^n \zeta_k) P_{\tau_n}. \end{aligned}$$

Together with (5.3.11), this implies that for all $n \geq 1$,

$$\sum_{k=1}^n |\zeta_k| P_{\tau_k} f(\zeta_k, \Theta_{\tau_k^-}) \ \leq \ x + (y + \sum_{k=1}^n |\zeta_k|) \sup_{s \in [t,T]} P^{t,p}_s + 2 \sum_{k=1}^n |\zeta_k| P_{\tau_k}.$$

The convergence of the series $\sum_n |\zeta_n| P_{\tau_n} f(\zeta_n, \Theta_{\tau_n^-})$ follows therefore from the convergence of the series $\sum_n |\zeta_n|$ and $\sum_n |\zeta_n| P_{\tau_n}$. □

As a consequence of the above results, we can now prove that in the optimal portfolio liquidation, it suffices to restrict to a finite number of trading times, which are strictly increasing. Given a trading strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 0} \in \mathcal{A}$, let us denote by $N(\alpha)$ the process counting the number of intervention times:

$$N_t(\alpha) \ = \ \sum_{n \geq 1} \mathbf{1}_{\tau_n \leq t}, \quad 0 \leq t \leq T.$$

We denote by $\mathcal{A}_\ell^b(t,z,\theta)$ the set of admissible trading strategies in $\mathcal{A}_\ell(t,z,\theta)$ with a finite number of trading times, such that these trading times are strictly increasing, namely:

$$\mathcal{A}_\ell^b(t,z,\theta) = \Big\{\alpha = (\tau_n,\zeta_n)_{n\geq 0} \in \mathcal{A}_\ell(t,z,\theta) : \; N_T(\alpha) < \infty, \quad a.s. \\ \text{and } \; \tau_n < \tau_{n+1} \; a.s., \quad 0 \leq n \leq N_T(\alpha) - 1\Big\}.$$

For any $\alpha = (\tau_n,\zeta_n)_n \in \mathcal{A}_\ell^b(t,z,\theta)$, the associated state process (Z,Θ) satisfies $\Theta_{\tau_{n+1}^-} > 0$, i.e. $(Z_{\tau_{n+1}^-},\Theta_{\tau_{n+1}^-}) \in \bar{\mathcal{S}}^* := \Big\{(z,\theta) \in \bar{\mathcal{S}} : \theta > 0\Big\}$. We also set $\partial_L \mathcal{S}^* = \partial_L \mathcal{S} \cap \bar{\mathcal{S}}^*$.

Theorem 5.3.1 *Assume that* **(H1f)**, **(H2f)**, **(H3f)**, **(Hcf)** *and* **(HU)** *hold. Then, we have*

$$v(t,z,\theta) = \sup_{\alpha\in\mathcal{A}_\ell^b(t,z,\theta)} \mathbb{E}[U(X_T)], \quad (t,z,\theta) \in [0,T]\times\bar{\mathcal{S}}. \tag{5.3.12}$$

Moreover, we have

$$v(t,z,\theta) = \sup_{\alpha\in\mathcal{A}_{\ell_+}^b(t,z,\theta)} \mathbb{E}[U(X_T)], \quad (t,z,\theta) \in [0,T]\times(\bar{\mathcal{S}}\setminus\partial_L\mathcal{S}), \tag{5.3.13}$$

where $\mathcal{A}_{\ell_+}^b(t,z,\theta) = \{\alpha \in \mathcal{A}_\ell^b(t,z,\theta) : (Z_s,\Theta_s) \in (\bar{\mathcal{S}}\setminus\partial_L\mathcal{S}), t \leq s < T\}$.

Proof. 1. Fix $(t,z,\theta) \in [0,T]\times\bar{\mathcal{S}}$, and denote by $\bar{\mathcal{A}}_\ell^b(t,z,\theta)$ the set of admissible trading strategies in $\mathcal{A}_\ell(t,z,\theta)$ with a finite number of trading times:

$$\bar{\mathcal{A}}_\ell^b(t,z,\theta) = \Big\{\alpha = (\tau_k,\zeta_k)_{k\geq 0} \in \mathcal{A}_\ell(t,z,\theta) : \; N_T(\alpha) \text{ is bounded a.s. } \Big\}.$$

Given an arbitrary $\alpha = (\tau_k,\zeta_k)_{k\geq 0} \in \mathcal{A}_\ell(t,z,\theta)$ associated to the state process $(Z,\Theta) = (X,Y,P,\Theta)$, let us consider the truncated trading strategy $\alpha^{(n)} = (\tau_k,\zeta_k)_{k\leq n} \cup (\tau_{n+1}, -Y_{\tau_{n+1}^-})$, which consists in liquidating all stock shares at time τ_{n+1}. This strategy $\alpha^{(n)}$ lies in $\bar{\mathcal{A}}_\ell(t,z,\theta)$, and is associated to the state process denoted by $(Z^{(n)},\Theta^{(n)})$. We then have

$$X_T^{(n)} - X_T = \sum_{k\geq n+1} \zeta_k P_{\tau_k} f(\zeta_k,\Theta_{\tau_k^-}) + Y_{\tau_{n+1}^-} P_{\tau_{n+1}} f(-Y_{\tau_{n+1}^-},\Theta_{\tau_{n+1}^-}).$$

Now, from Proposition 5.3.2, we have

$$\sum_{k\geq n+1} \zeta_k P_{\tau_k} f(\zeta_k,\Theta_{\tau_k^-}) \longrightarrow 0 \quad a.s. \quad \text{when} \quad n\to\infty.$$

Moreover, since $0 \leq Y_{\tau_{n+1}^-} = Y_{\tau_n}$ goes to $Y_T = 0$ as n goes to infinity, by definition of $\alpha \in \mathcal{A}_\ell(t,z,\theta)$, and recalling that f is smaller than 1 on $\mathbb{R}_-\times[0,T]$, we deduce that

$$0 \leq Y_{\tau_{n+1}^-} P_{\tau_{n+1}} f(-Y_{\tau_{n+1}^-}\Theta_{\tau_{n+1}^-}) \leq Y_{\tau_{n+1}^-} \sup_{s\in[t,T]} P_s^{t,p} \\ \longrightarrow 0 \quad a.s. \quad \text{when} \quad n\to\infty.$$

This proves that

$$X_T^{(n)} \longrightarrow X_T \quad a.s. \quad \text{when} \quad n \to \infty.$$

From Proposition 5.3.1, the sequence $(U(X_T^{(n)}))_{n\geq 1}$ is uniformly integrable, and we can apply the dominated convergence theorem to get

$$\mathbb{E}[U(X_T^{(n)})] \longrightarrow \mathbb{E}[U(X_T)], \qquad \text{when } n \to \infty.$$

From the arbitrariness of $\alpha \in \mathcal{A}_\ell(t,z,\theta)$, this shows that

$$v(t,z,\theta) \leq \bar{v}^b(t,z,\theta) := \sup_{\alpha\in\bar{\mathcal{A}}_\ell^b(t,z,\theta)} \mathbb{E}[U(X_T)],$$

and actually the equality $v = \bar{v}^b$ since the other inequality $\bar{v}^b \leq v$ is trivial from the inclusion $\bar{\mathcal{A}}_\ell^b(t,z,\theta) \subset \mathcal{A}_\ell(t,z,\theta)$.

2. Denote by v^b the value function in the r.h.s. of (5.3.12). It is clear that $v^b \leq \bar{v}^b = v$ since $\mathcal{A}_\ell^b(t,z,\theta) \subset \bar{\mathcal{A}}_\ell^b(t,z,\theta)$. To prove the reverse inequality we need first to study the behavior of optimal strategies at time T. Introduce the set

$$\tilde{\mathcal{A}}_\ell^b(t,z,\theta) = \Big\{\alpha = (\tau_k,\zeta_k)_k \in \mathcal{A}_\ell^b(t,z,\theta) \ : \ \#\{k \ : \ \tau_k = T\} \leq 1\Big\},$$

and denote by $\tilde{v}^b$ the associated value function. Then we have $\tilde{v}^b \leq \bar{v}^b$. Indeed, let $\alpha = (\tau_k,\zeta_k)_k$ be some arbitrary element in $\bar{\mathcal{A}}_\ell^b(t,z,\theta)$, $(t, z = (x,y,p),\theta) \in [0,T]\times\bar{\mathcal{S}}$. If $\alpha \in \tilde{\mathcal{A}}_\ell^b(t,z,\theta)$ then we have $\tilde{v}^b(t,z,\theta) \geq \mathbb{E}\Big[U_L(Z_T,\Theta_T)\Big]$, where (Z,Θ) denotes the process associated to α. Suppose now that $\alpha \notin \tilde{\mathcal{A}}_\ell^b(t,z,\theta)$. Set $m = \max\{k \ : \ \tau_k < T\}$. Then define the stopping time $\tau' := \frac{\tau_m+T}{2}$ and the $\mathcal{F}_{\tau'}$-measurable random variable $\zeta' := \mathrm{argmax}\{ef(e,T-\tau_m) \ : \ e \geq -Y_{\tau_m}\}$. Define the strategy $\alpha' = (\tau_k,\zeta_k)_{k\leq m} \cup (\tau', Y_{\tau_m} - \zeta') \cup (T,\zeta')$. From the construction of α', we easily check that $\alpha' \in \tilde{\mathcal{A}}^b(t,z,\theta)$ and $\mathbb{E}\Big[U_L(Z_T,\Theta_T)\Big] \leq \mathbb{E}\Big[U_L(Z'_T,\Theta'_T)\Big]$ where (Z',Θ') denotes the process associated to α'. Hence, we get $\tilde{v}^b \geq \bar{v}^b$.

We now prove that $v^b \geq \tilde{v}^b$. Let $\alpha = (\tau_k,\zeta_k)_k$ be some arbitrary element in $\tilde{\mathcal{A}}_\ell^b(t,z,\theta)$, $(t, z = (x,y,p),\theta) \in [0,T]\times\bar{\mathcal{S}}$. Denote by $N = N_T(\alpha)$ the a.s. finite number of trading times in α. We set $m = \inf\{0 \leq k \leq N-1 : \ \tau_{k+1} = \tau_k\}$ and $M = \sup\{m+1 \leq k \leq N : \ \tau_k = \tau_m\}$ with the convention that $\inf\emptyset = \sup\emptyset = N+1$. We then define $\alpha' = (\tau'_k,\zeta'_k)_{0\leq k\leq N-(M-m)+1} \in \mathcal{A}$ by:

$$(\tau'_k,\zeta'_k) = \begin{cases} (\tau_k,\zeta_k), & \text{for } 0 \leq k < m \\ (\tau_m = \tau_M, \sum_{k=m}^M \zeta_k), & \text{for } k = m \text{ and } m < N, \\ (\tau_{k+M-m},\zeta_{k+M-m}), & \text{for } m+1 \leq k \leq N-(M-m) \text{ and } m < N, \\ (\tau', \sum_{l=m+1}^M \zeta_l) & \text{for } k = N-(M-m)+1 \end{cases}$$

where $\tau' = \frac{\hat{\tau}+T}{2}$ with $\hat{\tau} = \max\{\tau_k \ : \ \tau_k < T\}$, and we denote by $(Z' = (X',Y',P),\Theta')$ the associated state process. It is clear that $(Z'_s,\Theta'_s) = (Z_s,\Theta_s)$ for $t \leq s < \tau_m$, and so $X'_{(\tau)'-}$

$= X_{(\tau')^-}$, $\Theta'_{(\tau')^-} = \Theta_{(\tau')^-}$. Moreover, since $\tau_m = \tau_M$, we have $\Theta_{\tau_k^-} = 0$ for $m+1 \leq k \leq M$. From Lemma 5.3.1 (or Remark 5.3.1), this implies that $\zeta_k \leq 0$ for $m+1 \leq k \leq M$, and so $\zeta'_{N-(M-m)+1} = \sum_{k=m+1}^{M} \zeta_k \leq 0$. We also recall that immediate sales does not increase the cash holdings, so that $X_{\tau_k} = X_{\tau_m}$ for $m+1 \leq k \leq M$. We then get

$$\begin{aligned} X'_T &= X_T - \zeta'_{N-(M-m)+1} P_{\tau'} f(\zeta'_{N-(M-m)+1}, \Theta'_{(\tau')^-}) \\ &\geq X_T. \end{aligned}$$

Moreover, we have $Y'_T = y + \sum_{k=1}^{N} \zeta_k = Y_T = 0$. By construction, notice that $\tau'_0 < \ldots < \tau'_{m+1}$. Given an arbitrary $\alpha \in \bar{\mathcal{A}}_\ell^b(t,z,\theta)$, we can then construct by induction a trading strategy $\alpha' \in \mathcal{A}_\ell^b(t,z,\theta)$ such that $X'_T \geq X_T$ a.s. By the nondecreasing monotonicity of the utility function U, this yields

$$\mathbb{E}[U(X_T)] \;\leq\; \mathbb{E}[U(X'_T)] \;\leq\; v^b(t,z,\theta),$$

and we conclude from the arbitrariness of $\alpha \in \tilde{\mathcal{A}}_\ell^b(t,z,\theta)$: $\tilde{v}^b \leq v^b$, and thus $v = \bar{v}^b = \tilde{v}^b = v^b$.

3. Fix now an element $(t,z,\theta) \in [0,T] \times (\bar{\mathcal{S}} \setminus \partial_L \mathcal{S})$, and denote by v_+ the r.h.s of (5.3.13). It is clear that $v \geq v_+$. Conversely, take some arbitrary $\alpha = (\tau_k, \zeta_k)_k \in \mathcal{A}_\ell^b(t,z,\theta)$, associated with the state process (Z,Θ), and denote by $N = N_T(\alpha)$ the finite number of trading times in α. Consider the first time before T when the liquidation value reaches zero, i.e. $\tau^\alpha = \inf\{t \leq s \leq T : L(Z_s,\Theta_s) = 0\} \wedge T$ with the convention $\inf\emptyset = \infty$. We claim that there exists $1 \leq m \leq N+1$ (depending on ω and α) such that $\tau^\alpha = \tau_m$, with the convention that $m = N+1$, $\tau_{N+1} = T$ if $\tau^\alpha = T$. On the contrary, there would exist $1 \leq k \leq N$ such that $\tau_k < \tau^\alpha < \tau_{k+1}$, and $L(Z_{\tau^\alpha}, \Theta_{\tau^\alpha}) = 0$. Between τ_k and τ_{k+1}, there is no trading, and so $(X_s,Y_s) = (X_{\tau_k}, Y_{\tau_k})$, $\Theta_s = s - \tau_k$ for $\tau_k \leq s < \tau_{k+1}$. We then get

$$L(Z_s,\Theta_s) \;=\; X_{\tau_k} + Y_{\tau_k} P_s f(-Y_{\tau_k}, s-\tau_k), \quad \tau_k \leq s < \tau_{k+1}. \tag{5.3.14}$$

Moreover, since $0 < L(Z_{\tau_k},\Theta_{\tau_k}) = X_{\tau_k}$, and $L(Z_{\tau^\alpha},\Theta_{\tau^\alpha}) = 0$, we see with (5.3.14) for $s = \tau^\alpha$ that $Y_{\tau_k} P_{\tau^\alpha} f(-Y_{\tau_k}, \tau^\alpha - \tau_k)$ should necessarily be strictly negative: $Y_{\tau_k} P_{\tau^\alpha} f(-Y_{\tau_k}, \tau^\alpha - \tau_k) < 0$, a contradiction with the admissibility conditions and the nonnegative property of f.

We then have $\tau^\alpha = \tau_m$ for some $1 \leq m \leq N+1$. Observe that if $m \leq N$, i.e. $L(Z_{\tau_m},\Theta_{\tau_m}) = 0$, then $U(L(Z_T,\Theta_T)) = 0$. Indeed, suppose that $Y_{\tau_m} > 0$ and $m \leq N$. From the admissibility condition, and by Itô's formula to $L(Z,\Theta)$ in (5.3.14) between τ^α and τ_{m+1}^-, we get

$$\begin{aligned} 0 \;\leq\; L(Z_{\tau_{m+1}^-}, \Theta_{\tau_{k+1}^-}) &= L(Z_{\tau_{m+1}^-}, \Theta_{\tau_{m+1}^-}) - L(Z_{\tau^\alpha}, \Theta_{\tau^\alpha}) \\ &= \int_{\tau^\alpha}^{\tau_{m+1}} Y_{\tau_m} P_s \Big[\beta(Y_{\tau_m}, s-\tau_m)ds + \sigma f(-Y_{\tau_k}, s-\tau_m) dW_s\Big] \end{aligned} \tag{5.3.15}$$

where $\beta(y,\theta) = bf(-y,\theta) + \dfrac{\partial f}{\partial\theta}(-y,\theta)$ is bounded on $\mathbb{R}_+ \times [0,T]$ by **(Hcf)**(ii). Since the integrand in the above stochastic integral w.r.t Brownian motion W is strictly positive, thus nonzero, we must have $\tau^\alpha = \tau_{m+1}$. Otherwise, there is a nonzero probability that the r.h.s. of (5.3.15) becomes strictly negative, a contradiction with the inequality (5.3.15).

Hence we get $Y_{\tau_m} = 0$, and thus $L(Z_{\tau_{m+1}^-}, \Theta_{\tau_{m+1}^-}) = X_{\tau_m} = 0$. From the Markov feature of the model and Corollary 5.3.1, we then have

$$\mathbb{E}\Big[U\Big(L(Z_T, \Theta_T)\Big)\Big|\mathcal{F}_{\tau_m}\Big] \;\leq\; v(\tau_m, Z_{\tau_m}, \Theta_{\tau_m}) \;=\; U(X_{\tau_m}) \;=\; 0.$$

Since U is nonnegative, this implies that $U(L(Z_T, \Theta_T)) = 0$. Let us next consider the trading strategy $\alpha' = (\tau'_k, \zeta'_k)_{0\leq k\leq(m-1)} \in \mathcal{A}$ consisting in following α until time τ^α, and liquidating all stock shares at time $\tau^\alpha = \tau_{m-1}$, and defined by:

$$(\tau'_k, \zeta'_k) \;=\; \begin{cases} (\tau_k, \zeta_k), & \text{for } 0 \leq k < m-1 \\ \Big(\tau_{m-1}, -Y_{\tau^-_{(m-1)}}\Big), & \text{for } k = m-1, \end{cases}$$

and we denote by (Z', Θ') the associated state process. It is clear that $(Z'_s, \Theta'_s) = (Z_s, \Theta_s)$ for $t \leq s < \tau_{m-1}$, and so $L(Z'_s, \Theta'_s) = L(Z_s, \Theta_s) > 0$ for $t \leq s \leq \tau_{m-1}$. The liquidation at time τ_{m-1} (for $m \leq N$) yields $X_{\tau_{m-1}} = L(Z_{\tau_{m-1}^-}, \Theta_{\tau_{m-1}^-}) > 0$, and $Y_{\tau_{m-1}} = 0$. Since there is no more trading after time τ_{m-1}, the liquidation value for $\tau_{m-1} \leq s \leq T$ is given by: $L(Z_s, \Theta_s) = X_{\tau_{m-1}} > 0$. This shows that $\alpha' \in \mathcal{A}^b_{\ell_+}(t, z, \theta)$. When $m = N+1$, we have $\alpha = \alpha'$, and so $X'_T = L(Z'_T, \Theta'_T) = L(Z_T, \Theta_T) = X_T$. For $m \leq N$, we have $U(X'_T) = U(L(Z'_T, \Theta'_T)) \geq 0 = U(L(Z_T, \Theta_T)) = U(X_T)$. We then get $U(X'_T) \geq U(X_T)$ a.s., and so

$$\mathbb{E}[U(X_T)] \;\leq\; \mathbb{E}[U(X'_T)] \;\leq\; v_+(t, z, \theta).$$

We conclude from the arbitrariness of $\alpha \in \bar{\mathcal{A}}^b_\ell(t, z, \theta)$: $v \leq v_+$, and thus $v = v_+$. □

Remark 5.3.2 If we suppose that the function $e \in \mathbb{R} \mapsto ef(e, \theta)$ is increasing for $\theta \in (0, T]$, we get the value of v on the bound $\partial_L\mathcal{S}^*$: $v(t, z, \theta) = U(0) = 0$ for $(t, z = (x, y, p), \theta) \in [0, T] \times \partial_L\mathcal{S}^*$. Indeed, fix some point $(t, z = (x, y, p), \theta) \in [0, T] \times \partial_L\mathcal{S}^*$, and consider an arbitrary $\alpha = (\tau_k, \zeta_k)_k \in \mathcal{A}^b_\ell(t, z, \theta)$ with state process (Z, Θ), and denote by N the number of trading times. We distinguish two cases: (i) If $\tau_1 = t$, then by Lemma 5.3.1, the transaction ζ_1 is equal to $-y$, which leads to $Y_{\tau_1} = 0$, and a liquidation value $L(Z_{\tau_1}, \Theta_{\tau_1})$ $= X_{\tau_1} = L(z, \theta) = 0$. At the next trading date τ_2 (if it exists), we get $X_{\tau_2^-} = Y_{\tau_2^-} = 0$ with liquidation value $L(Z_{\tau_2^-}, \Theta_{\tau_2^-}) = 0$, and by using again Lemma 5.3.1, we see that after the transaction at τ_2, we shall also obtain $X_{\tau_2} = Y_{\tau_2} = 0$. By induction, this leads at the final trading time to $X_{\tau_N} = Y_{\tau_N} = 0$, and finally to $X_T = Y_T = 0$. (ii) If $\tau_1 > t$, we claim that $y = 0$. On the contrary, by arguing similarly as in (5.3.15) between t and τ_1^-, we have then proved that any admissible trading strategy $\alpha \in \mathcal{A}^b_\ell(t, z, \theta)$ provides a final liquidation value $X_T = 0$, and so

$$v(t, z, \theta) \;=\; U(0) = 0, \qquad \forall (t, z, \theta) \in [0, T] \times \partial_L\mathcal{S}^*. \tag{5.3.16}$$

Remark 5.3.3 The representation (5.3.12) of the optimal portfolio liquidation reveals interesting economical and mathematical features. It shows that the liquidation problem in a continuous-time illiquid market model with discrete-time orders and temporary price impact with the presence of a bid-ask spread as considered in this paper, leads to nearly optimal trading strategies with a finite number of orders and with strictly increasing trading times. While most models dealing with trading strategies via an impulse control formulation assumed fixed transaction fees in order to justify the discrete nature of trading times, we prove in this paper that discrete-time trading appears naturally as a consequence of temporary price impact and bid-ask spread.

The representation (5.3.13) shows that when we are in an initial state with strictly positive liquidation value, then we can restrict in the optimal portfolio liquidation problem to admissible trading strategies with strictly positive liquidation value up to time T^-. The relation (5.3.16) means that when the initial state has a zero liquidation value, which is not a result of an immediate trading time, then the liquidation value will stay at zero until the final horizon.

5.4 Dynamic programming and viscosity properties

In the sequel, the conditions **(H1f)**, **(H2f)**, **(H3f)**, **(Hcf)** and **(HU)** stand in force, and are not recalled in the statement of Theorems and Propositions.

We use a dynamic programming approach to derive the equation satisfied by the value function of our optimal portfolio liquidation problem. Dynamic programming principle (DPP) for impulse controls was frequently used starting from the works by Bensoussan and Lions [8], and then considered e.g. in [78], [60], [53] or [75]. In our context (recall the expression (5.2.10) of the value function), this is formulated as:

Dynamic programming principle (DPP). For all $(t, z, \theta) \in [0, T] \times \bar{\mathcal{S}}$, we have

$$v(t, z, \theta) \quad = \quad \sup_{\alpha \in \mathcal{A}(t,z,\theta)} \mathbb{E}[v(\tau, Z_\tau, \Theta_\tau)], \tag{5.4.1}$$

where $\tau = \tau(\alpha)$ is any stopping time valued in $[t, T]$ eventually depending on the strategy α in (5.4.1). More precisely we have :

(i) for all $\alpha \in \mathcal{A}(t, z, \theta)$, for all $\tau \in \mathcal{T}_{t,T}$, the set of stopping times valued in $[t, T]$:

$$\mathbb{E}[v(\tau, Z_\tau, \Theta_\tau)] \quad \leq \quad v(t, z, \theta) \tag{5.4.2}$$

(ii) for all $\varepsilon > 0$, there exists $\hat{\alpha}^\varepsilon \in \mathcal{A}(t, z, \theta)$ s.t. for all $\tau \in \mathcal{T}_{t,T}$:

$$v(t, z, \theta) - \varepsilon \quad \leq \quad \mathbb{E}[v(\tau, \hat{Z}^\varepsilon_\tau, \hat{\Theta}^\varepsilon_\tau)], \tag{5.4.3}$$

with $(\hat{Z}^\varepsilon, \hat{\Theta}^\varepsilon)$ the state process controlled by $\hat{\alpha}^\varepsilon$.

The corresponding dynamic programming Hamilton-Jacobi-Bellman (HJB) equation is a quasi-variational inequality (QVI) written as:

$$\min \Big[- \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \theta} - \mathcal{L} v \; , \; v - \mathcal{H} v \Big] \;\; = \;\; 0, \quad \text{ in } [0,T) \times \bar{\mathcal{S}}, \tag{5.4.4}$$

together with the relaxed terminal condition:

$$\min \left[v - U_L \; , \; v - \mathcal{H} v \right] \;\; = \;\; 0, \quad \text{ in } \{T\} \times \bar{\mathcal{S}}. \tag{5.4.5}$$

The rigorous derivation of the HJB equation satisfied by the value function from the dynamic programming principle is achieved by means of the notion of viscosity solutions, and is by now rather classical in the modern approach of stochastic control (see e.g. the books [34] and [67]). There are some specificities here related to the impulse control and the liquidation state constraint, and we recall in Appendix, definitions of (discontinuous) constrained viscosity solutions for parabolic QVIs. The main result of this section is stated as follows.

Theorem 5.4.1 *The value function v is a constrained viscosity solution to* (5.4.4)-(5.4.5).

Proof. The proof of the viscosity supersolution property on $[0,T) \times \mathcal{S}$ and the viscosity subsolution property on $[0,T) \times \bar{\mathcal{S}}$ follows the same lines of arguments as in [53], and is then omitted here. We focus on the terminal condition (5.4.5).

We first check the viscosity supersolution property on $\{T\} \times \mathcal{S}$. Fix some $(z,\theta) \in \mathcal{S}$, and consider some sequence $(t_k, z_k, \theta_k)_{k \geq 1}$ in $[0,T) \times \mathcal{S}$, converging to (T,z,θ) and such that $\lim_k v(t_k, z_k, \theta_k) = v_*(T,z,\theta)$. By taking the no impulse control strategy on $[t_k, T]$, we have

$$v(t_k, z_k, \theta_k) \;\; \geq \;\; \mathbb{E}[U_L(Z_T^{0,t_k,z_k}, \Theta_T^{0,t_k,\theta_k})].$$

Since $(Z_T^{0,t_k,z_k}, \Theta_T^{0,t_k,z_k})$ converges a.s. to (z,θ) when k goes to infinity by continuity of $(Z^{0,t,z}, \Theta^{0,t,\theta})$ in its initial condition, we deduce by Fatou's lemma that

$$v_*(T,z,\theta) \;\; \geq \;\; U_L(z,\theta). \tag{5.4.6}$$

On the other hand, we know from the dynamic programming QVI that $v \geq \mathcal{H} v$ on $[0,T) \times \mathcal{S}$, and thus

$$v(t_k, z_k, \theta_k) \;\; \geq \;\; \mathcal{H} v(t_k, z_k, \theta_k) \; \geq \; \mathcal{H} v_*(t_k, z_k, \theta_k), \quad \forall k \geq 1.$$

Recalling that $\mathcal{H} v_*$ is lsc, we obtain by sending k to infnity:

$$v_*(T,z,\theta) \;\; \geq \;\; \mathcal{H} v_*(T,z,\theta).$$

Together with (5.4.6), this proves the required viscosity supersolution property of (5.4.5).

We now prove the viscosity subsolution property on $\{T\}\times\bar{\mathcal{S}}$, and argue by contradiction by assuming that there exists $(\bar z,\bar\theta)\in\bar{\mathcal{S}}$ such that

$$\min\left[v^*(T,\bar z,\bar\theta)-U_L(\bar z,\bar\theta)\ ,\ v^*(T,\bar z,\bar\theta)-\mathcal{H}v^*(T,\bar z,\bar\theta)\right] \ := \ 2\varepsilon \ > \ 0. \tag{5.4.7}$$

One can find a sequence of smooth functions $(\varphi^n)_{n\geq 0}$ on $[0,T]\times\bar{\mathcal{S}}$ such that φ^n converges pointwisely to v^* on $[0,T]\times\bar{\mathcal{S}}$ as $n\to\infty$. Moreover, by (5.4.7) and recalling that $\mathcal{H}v^*$ is usc, we may assume that the inequality

$$\min\left[\varphi^n-U_L\ ,\ \varphi^n-\mathcal{H}v^*\right] \ \geq \ \varepsilon, \tag{5.4.8}$$

holds on some bounded neighborhood B^n of $(T,\bar z,\bar\theta)$ in $[0,T]\times\bar{\mathcal{S}}$, for n large enough. Let $(t_k,z_k,\theta_k)_{k\geq 1}$ be a sequence in $[0,T)\times\mathcal{S}$ converging to $(T,\bar z,\bar\theta)$ and such that $\lim_k v(t_k,z_k,\theta_k)$ $=v^*(T,\bar z,\bar\theta)$. There exists $\delta^n>0$ such that $B^n_k:=[t_k,T]\times B(z_k,\delta^n)\times\Big((\theta_k-\delta^n,\theta_k+\delta^n)\cap[0,T]\Big)\subset B^n$ for all k large enough, so that (5.4.8) holds on B^n_k. Since v is locally bounded, there exists some $\eta>0$ such that $|v^*|\leq\eta$ on B^n. We can then assume that $\varphi^n\geq -2\eta$ on B^n. Let us define the smooth function $\tilde\varphi^n_k$ on $[0,T)\times\mathcal{S}$ by

$$\tilde\varphi^n_k(t,z,\theta) \ := \ \varphi^n(t,z,\theta)+4\eta\frac{|z-z_k|^2}{|\delta^n|^2}+\sqrt{T-t}$$

and observe that

$$(v^*-\tilde\varphi^n_k)(t,z,\theta) \ \leq \ -\eta, \tag{5.4.9}$$

for $(t,z,\theta)\in[t_k,T]\times\partial B(z_k,\delta^n)\times\Big((\theta_k-\delta^n,\theta_k+\delta^n)\cap[0,T]\Big)$. Since $\dfrac{\partial\sqrt{T-t}}{\partial t}\longrightarrow-\infty$ as $t\to T$, we have for k large enough

$$-\frac{\partial\tilde\varphi^n_k}{\partial t}-\frac{\partial\tilde\varphi^n_k}{\partial\theta}-\mathcal{L}\tilde\varphi^n_k \ \geq \ 0, \quad \text{on} \ \ B^n_k. \tag{5.4.10}$$

Let $\alpha^k=(\tau^k_j,\zeta^k_j)_{j\geq 1}$ be a $\frac{1}{k}-$optimal control for $v(t_k,z_k,\theta_k)$ with corresponding state process (Z^k,Θ^k), and denote by $\sigma^k_n=\inf\{s\geq t_k\ :\ (Z^k_s,\Theta^k_s)\notin B^n_k\}\wedge\tau^k_1\wedge T$. From the DPP (5.4.3), this means that

$$\begin{aligned}
v(t_k,z_k,\theta_k)-\frac{1}{k} \ \leq \ & \mathbb{E}\Big[\mathbf{1}_{\sigma^k_n<(\tau^k_1\wedge T)}\,v(\sigma^k_n,Z^k_{\sigma^k_n})\Big]+\mathbb{E}\Big[\mathbf{1}_{\sigma^k_n=T<\tau^k_1}\,U_L(Z^k_{\sigma^k_n},\Theta^k_{\sigma^k_n})\Big]\\
&+\mathbb{E}\Big[\mathbf{1}_{\tau^k_1\leq\sigma^k_n}\,v(\tau^k_1,\Gamma(Z^k_{(\tau^k_1)^-},\Theta^k_{(\tau^k_1)^-},\zeta^k_1),0)\Big]
\end{aligned} \tag{5.4.11}$$

Now, by applying Itô's Lemma to $\tilde\varphi^k_n(s,Z^k_s,\Theta^k_s)$ between t_k and σ^k_n, we get from (5.4.8)-(5.4.9)-(5.4.10),

$$\begin{aligned}
\tilde\varphi^n_k(t_k,z_k,\theta_k) \ \geq \ & \mathbb{E}\Big[\mathbf{1}_{\sigma^k_n<\tau^k_1}\,\tilde\varphi^n_k(\sigma^k_n,Z^k_{\sigma^k_n},\Theta^k_{\sigma^k_n})\Big]+\mathbb{E}\Big[\mathbf{1}_{\tau^k_1\leq\sigma^k_n}\tilde\varphi^n_k(\tau^k_1,Z^k_{(\tau^k_1)^-},\Theta^k_{(\tau^k_1)^-})\Big]\\
\geq \ & \mathbb{E}\Big[\mathbf{1}_{\sigma^k_n<(\tau^k_1\wedge T)}\Big(v^*(\sigma^k_n,Z^k_{\sigma^k_n},\Theta^k_{\sigma^k_n})+\eta\Big)\Big]+\mathbb{E}\Big[\mathbf{1}_{\sigma^k_n=T<\tau^k_1}\Big(U_L(Z^k_{\sigma^k_n},\Theta^k_{\sigma^k_n})+\varepsilon\Big)\Big]\\
&+\mathbb{E}\Big[\mathbf{1}_{\tau^k_1\leq\sigma^k_n}\Big(v^*(\tau^k_1,\Gamma(Z^k_{(\tau^k_1)^-},\Theta^k_{(\tau^k_1)^-},\zeta^k_1),0)+\varepsilon\Big)\Big].
\end{aligned}$$

Together with (5.4.11), this implies

$$\tilde{\varphi}_k^n(t_k, z_k, \theta_k) \quad \geq \quad v(t_k, z_k, \theta_k) - \frac{1}{k} + \varepsilon \wedge \eta.$$

Sending k, and then n to infinity, we get the required contradiction: $v^*(T, \bar{z}, \bar{\theta}) \geq v^*(T, \bar{z}, \bar{\theta}) + \varepsilon \wedge \eta$. □

Remark 5.4.1 In order to have a complete characterization of the value function through its HJB equation, we need a uniqueness result, thus a comparison principle for the QVI (5.4.4)-(5.4.5). A key argument originally due to Ishii [45] for getting a uniqueness result for variational inequalities with impulse parts, is to produce a strict viscosity supersolution. However, in our model, this is not possible. Indeed, suppose we can find a strict viscosity lsc supersolution w to (5.4.4), so that $(w - \mathcal{H}w)(t, z, \theta) > 0$ on $[0, T) \times \mathcal{S}$. But for $z = (x, y, p)$ and $\theta = 0$, we have $\Gamma(z, 0, e) = (x, y + e, p)$ for any e $\mathcal{C}(z, 0)$. Since $0 \in \mathcal{C}(z, 0)$ we have $\mathcal{H}w(t, z, 0) = \sup_{e \in [-y, 0]} w(t, x, y + e, p, 0) \geq w(t, z, 0) > \mathcal{H}w(t, z, 0)$, a contradiction. Actually, the main reason why one cannot obtain a strict supersolution is the absence of fixed cost in the impulse function Γ or in the objective functional.

5.5 An approximating problem with fixed transaction fee

In this section, we consider a small variation of our original model by adding a fixed transaction fee $\varepsilon > 0$ at each trading. This means that given a trading strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 0}$, the controlled state process $(Z = (X, Y, P), \Theta)$ jumps now at time τ_{n+1}, by:

$$(Z_{\tau_{n+1}}, \Theta_{\tau_{n+1}}) \quad = \quad \Big(\Gamma_\varepsilon(Z_{\tau_{n+1}^-}, \Theta_{\tau_{n+1}^-}, \zeta_{n+1}), 0\Big), \tag{5.5.1}$$

where Γ_ε is the function defined on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0, T] \times \mathbb{R}$ into $\mathbb{R} \cup \{-\infty\} \times \mathbb{R} \times \mathbb{R}_+^*$ by:

$$\Gamma_\varepsilon(z, \theta, e) \quad = \quad \Gamma(z, \theta, e) - (\varepsilon, 0, 0) \;=\; \Big(x - epf(e, \theta) - \varepsilon, y + e, p\Big),$$

for $z = (x, y, p)$. The dynamics of (Z, Θ) between trading dates is given as before. We also introduce a modified liquidation function L_ε defined by:

$$L_\varepsilon(z, \theta) \quad = \quad \max[x, L(z, \theta) - \varepsilon], \quad (z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0, T].$$

The interpretation of this modified liquidation function is the following. Due to the presence of the transaction fee at each trading, it may be advantageous for the investor not to liquidate his position in stock shares (which would give him $L(z, \theta) - \varepsilon$), and rather bin his stock shares, by keeping only his cash amount (which would give him x). Hence, the investor chooses the best of these two possibilities, which induces a liquidation value $L_\varepsilon(z, \theta)$.

We then introduce the corresponding solvency region $\mathcal{S}_\varepsilon$ with its closure $\bar{\mathcal{S}}_\varepsilon = \mathcal{S}_\varepsilon \cup \partial\mathcal{S}_\varepsilon$, and boundary $\partial\mathcal{S}_\varepsilon = \partial_y\mathcal{S}_\varepsilon \cup \partial_L\mathcal{S}_\varepsilon$:

$$\mathcal{S}_\varepsilon \quad = \quad \Big\{(z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0, T] : \; y > 0 \; \text{ and } L_\varepsilon(z, \theta) > 0\Big\},$$

$$\begin{aligned}
\partial_y \mathcal{S}_\varepsilon &= \Big\{(z,\theta) = (x,y,p,\theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times [0,T] : \ y = 0 \ \text{ and } \ L_\varepsilon(z,\theta) \geq 0\Big\}, \\
\partial_L \mathcal{S}_\varepsilon &= \Big\{(z,\theta) = (x,y,p,\theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+ : \ L_\varepsilon(z,\theta) = 0\Big\}.
\end{aligned}$$

We also introduce the corner lines of $\partial \mathcal{S}_\varepsilon$. For simplicity of presentation, we consider a temporary price impact function f in the form:

$$f(e,\theta) \;=\; \tilde{f}\Big(\frac{e}{\theta}\Big) \;=\; \exp\Big(\lambda \frac{e}{\theta}\Big)\Big(\kappa_a \mathbf{1}_{e>0} + \mathbf{1}_{e=0} + \kappa_b \mathbf{1}_{e<0}\Big)\mathbf{1}_{\theta>0},$$

where $0 < \kappa_b < 1 < \kappa_a$, and $\lambda > 0$. A straightforward analysis of the function L shows that $y \mapsto L(x,y,p,\theta)$ is increasing on $[0,\theta/\lambda]$, decreasing on $[\theta/\lambda,\infty)$ with $L(x,0,p,\theta) = x = L(x,\infty,p,\theta)$, and $\max_{y>0} L(x,y,p,\theta) = L(x,\theta/\lambda,p,\theta) = x + p\frac{\theta}{\lambda}\tilde{f}(-1/\lambda)$. We first get the form of the sets $\mathcal{C}(z,\theta)$:

$$\mathcal{C}(z,\theta) \;=\; [-y, \bar{e}(z,\theta)]\,,$$

where the function $\bar{e}$ is defined in Lemma 5.3.1. We then distinguish two cases: (i) If $p\frac{\theta}{\lambda}\tilde{f}(-1/\lambda) < \varepsilon$, then $L_\varepsilon(x,y,p,\theta) = x$. (ii) If $p\frac{\theta}{\lambda}\tilde{f}(-1/\lambda) \geq \varepsilon$, then there exists an unique $y_1(p,\theta) \in (0,\theta/\lambda]$ and $y_2(p,\theta) \in [\theta/\lambda,\infty)$ such that $L(x,y_1(p,\theta),p,\theta) = L(x,y_2(p,\theta),p,\theta)$ $= x$, and $L_\varepsilon(x,y,p,\theta) = x$ for $y \in [0,y_1(p,\theta)) \cup (y_2(p,\theta),\infty)$, $L_\varepsilon(x,y,p,\theta) = L(x,y,p,\theta) - \varepsilon$ for $y \in [y_1(p,\theta), y_2(p,\theta)]$. We then denote by

$$\begin{aligned}
D_0 &= \{0\} \times \{0\} \times \mathbb{R}_+^* \times [0,T] \;=\; \partial_y \mathcal{S}_\varepsilon \cap \partial_L \mathcal{S}_\varepsilon, \\
D_{1,\varepsilon} &= \Big\{(0, y_1(p,\theta), p, \theta) : p\frac{\theta}{\lambda}\tilde{f}(\frac{-1}{\lambda}) \geq \varepsilon,\ \theta \in [0,T]\Big\}, \\
D_{2,\varepsilon} &= \Big\{(0, y_2(p,\theta), p, \theta) : p\frac{\theta}{\lambda}\tilde{f}(\frac{-1}{\lambda}) \geq \varepsilon,\ \theta \in [0,T]\Big\}.
\end{aligned}$$

Notice that the inner normal vectors at the corner lines $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$ form an acute angle (positive scalar product), while we have a right angle at the corner D_0.

Next, we define the set of admissible trading strategies as follows. Given $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}_\varepsilon$, we say that the impulse control α is admissible, denoted by $\alpha \in \mathcal{A}^\varepsilon(t,z,\theta)$, if τ_0 $= t - \theta$, $\tau_n \geq t$, $n \geq 1$, and the controlled state process (Z^ε, Θ) solution to (5.2.1)-(5.2.2)-(5.2.3)-(5.2.6)-(5.5.1), with an initial state $(Z^\varepsilon_{t^-}, \Theta_{t^-}) = (z,\theta)$ (and the convention that $(Z^\varepsilon_t, \Theta_t) = (z,\theta)$ if $\tau_1 > t$), satisfies $(Z^\varepsilon_s, \Theta_s) \in [0,T] \times \bar{\mathcal{S}}_\varepsilon$ for all $s \in [t,T]$. Here, we stress the dependence of $Z^\varepsilon = (X^\varepsilon, Y, P)$ in ε appearing in the transaction function Γ_ε, and we notice that it affects only the cash component. Notice that $\mathcal{A}^\varepsilon(t,z,\theta)$ is nonempty for any $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}_\varepsilon$. Indeed, for $(z = (x,y,p),\theta) \in \bar{\mathcal{S}}_\varepsilon$, i.e. $L_\varepsilon(z,\theta) = \max(x, L(z,\theta) - \varepsilon) \geq 0$, we distinguish two cases: (i) if $x \geq 0$, then by doing none transaction, the associated state process $(Z^\varepsilon = (X^\varepsilon, Y, P), \Theta)$ satisfies $X^\varepsilon_s = x \geq 0$, $t \leq s \leq T$, and thus this zero transaction is admissible; (ii) if $L(z,\theta) - \varepsilon \geq 0$, then by liquidating immediately all the stock shares, and doing nothing more after, the associated state process satisfies $X^\varepsilon_s = L(z,\theta) - \varepsilon$, $Y_s = 0$, and thus $L_\varepsilon(Z^\varepsilon_s, \Theta_s) = X^\varepsilon_s \geq 0$, $t \leq s \leq T$, which shows that this immediate transaction is admissible.

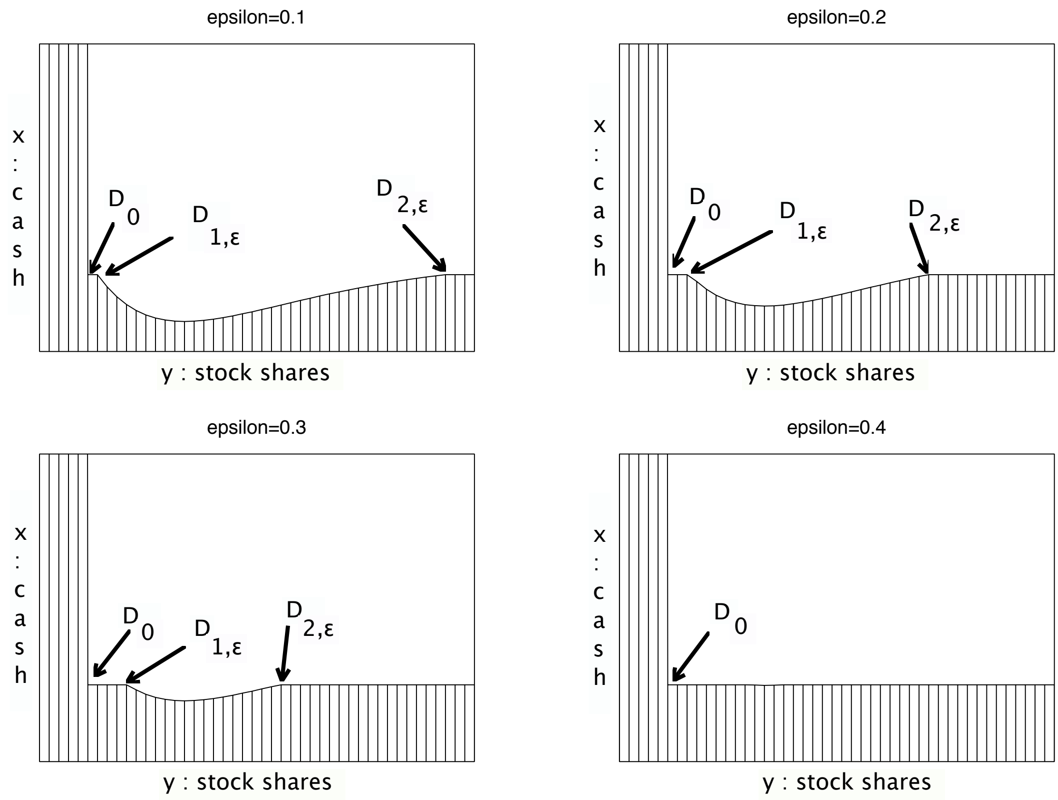

Figure 5.4: Domain $\mathcal{S}_\varepsilon$ in the nonhatched zone for fixed $p = 1$ and $\theta = 1$ and ε evolving from 0.1 to 0.4. Here $\kappa_b = 0.9$ and $f(e, \theta) = \kappa_b \exp\left(\frac{e}{\theta}\right)$ for $e < 0$. Notice that for ε large enough, $\mathcal{S}_\varepsilon$ is equal to open orthant $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

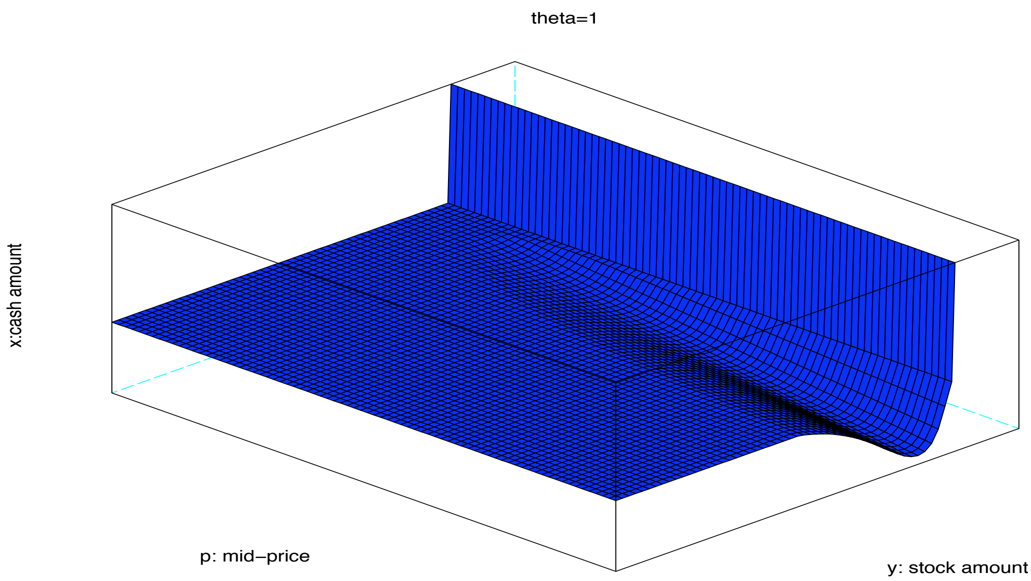

Figure 5.5: Lower bound of the domain $\mathcal{S}_\varepsilon$ for fixed $\theta = 1$ and $f(e, \theta) = \kappa_b \exp\left(\frac{e}{\theta}\right)$ for $e < 0$. Notice that when p is fixed, we obtain the Figure 4.

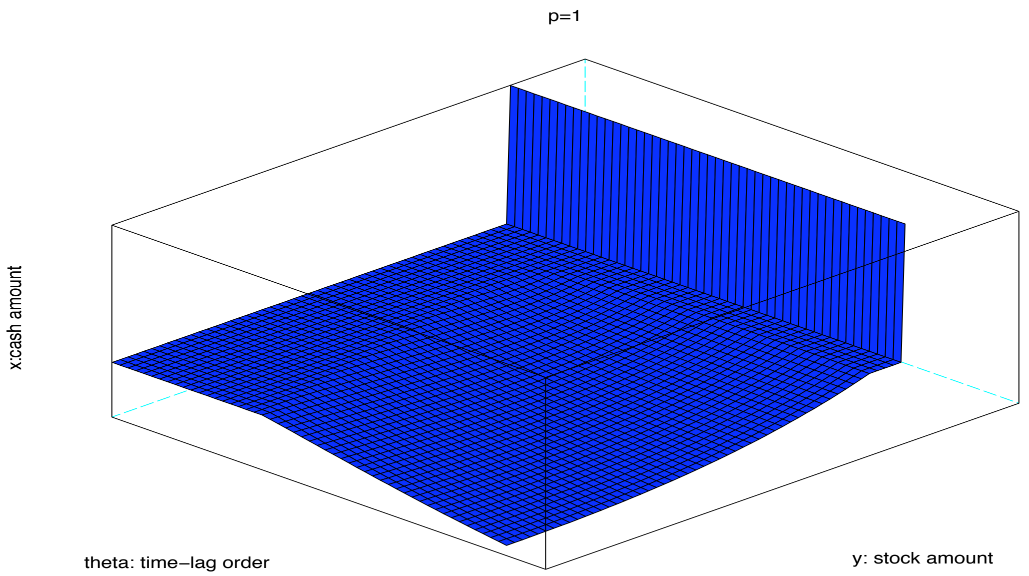

Figure 5.6: Lower bound of the domain $\mathcal{S}_\varepsilon$ for fixed $p = 1$ and $\varepsilon = 0.2$. Here $\kappa_b = 0.9$ and $f(e, \theta) = \kappa_b \exp\left(\frac{e}{\theta}\right)$ for $e < 0$. Notice that when θ is fixed, we obtain the Figure 4.

Given the utility function U on $\mathbb{R}_+$, and the liquidation utility function defined on $\bar{\mathcal{S}}_\varepsilon$ by $U_{L_\varepsilon}(z,\theta) = U(L_\varepsilon(z,\theta))$, we then consider the associated optimal portfolio liquidation problem defined via its value function by:

$$v_\varepsilon(t,z,\theta) \;=\; \sup_{\alpha\in\mathcal{A}^\varepsilon(t,z,\theta)} \mathbb{E}[U_{L_\varepsilon}(Z_T^\varepsilon,\Theta_T)], \quad (t,z,\theta)\in[0,T]\times\bar{\mathcal{S}}_\varepsilon. \tag{5.5.2}$$

Notice that when $\varepsilon = 0$, the above problem reduces to the optimal portfolio liquidation problem described in Section 2, and in particular $v_0 = v$. The main purpose of this section is to provide a unique PDE characterization of the value functions v_ε, $\varepsilon > 0$, and to prove that the sequence $(v_\varepsilon)_\varepsilon$ converges to the original value function v as ε goes to zero.

We define the set of admissible transactions in the model with fixed transaction fee by:

$$\mathcal{C}_\varepsilon(z,\theta) \;=\; \Big\{e\in\mathbb{R} : \Big(\Gamma_\varepsilon(z,\theta,e),0\Big)\in\bar{\mathcal{S}}_\varepsilon\Big\}, \quad (z,\theta)\in\bar{\mathcal{S}}_\varepsilon.$$

A similar calculation as in Lemma 5.3.1 shows that for $(z=(x,y,p),\theta)\in\bar{\mathcal{S}}_\varepsilon$,

$$\mathcal{C}_\varepsilon(z,\theta) \;=\; \begin{cases} [-y,\bar{e}_\varepsilon(z,\theta)], & \text{if } \theta>0 \text{ or } x\geq\varepsilon,\\ \emptyset, & \text{if } \theta=0 \text{ and } x<\varepsilon,\end{cases}$$

where $\bar{e}(z,\theta) = \sup\{e\in\mathbb{R} : ep\tilde{f}(e/\theta)\leq x-\varepsilon\}$ if $\theta>0$ and $\bar{e}(z,0)=0$ if $x\geq\varepsilon$. Here, the set $[-y,\bar{e}_\varepsilon(z,\theta)]$ should be viewed as empty when $\bar{e}(z,\theta)<y$, i.e. $x+py\tilde{f}(-y/\theta)-\varepsilon<0$. We also easily check that $\mathcal{C}_\varepsilon$ is continuous for the Hausdorff metric. We then consider the impulse operator $\mathcal{H}_\varepsilon$ by

$$\mathcal{H}_\varepsilon w(t,z,\theta) \;=\; \sup_{e\in\mathcal{C}_\varepsilon(z,\theta)} w(t,\Gamma_\varepsilon(z,\theta,e),0), \quad (t,z,\theta)\in[0,T]\times\bar{\mathcal{S}}_\varepsilon,$$

for any locally bounded function w on $[0,T]\times\bar{\mathcal{S}}_\varepsilon$, with the convention that $\mathcal{H}_\varepsilon w(t,z,\theta) = -\infty$ when $\mathcal{C}_\varepsilon(z,\theta)=\emptyset$.

Next, consider again the Merton liquidation function L_M, and observe similarly as in (5.3.7) that

$$\begin{aligned} L_M(\Gamma_\varepsilon(z,\theta,e)) - L_M(z) \;&=\; ep\Big(1-f(e,\theta)\Big)-\varepsilon \\ &\leq\; -\varepsilon, \quad \forall (z,\theta)\in\bar{\mathcal{S}}_\varepsilon,\;\; e\in\mathbb{R}. \end{aligned} \tag{5.5.3}$$

This implies in particular that

$$\mathcal{H}_\varepsilon L_M \;<\; L_M \quad \text{on } \bar{\mathcal{S}}_\varepsilon. \tag{5.5.4}$$

Since $L_\varepsilon\leq L_M$, we observe from (5.5.3) that if $(z,\theta)\in\mathcal{N}_\varepsilon := \{(z,\theta)\in\bar{\mathcal{S}}_\varepsilon : L_M(z)<\varepsilon\}$, then $\mathcal{C}_\varepsilon(z,\theta)=\emptyset$. Moreover, we deduce from (5.5.3) that for all $\alpha=(\tau_n,\zeta_n)_{n\geq0}\in\mathcal{A}^\varepsilon(t,z,\theta)$ associated to the state process (Z,Θ), $(t,z,\theta)\in[0,T]\times\bar{\mathcal{S}}_\varepsilon$:

$$\begin{aligned} 0 \;\leq\; L_M(Z_T) \;&=\; L_M(Z_T^{0,t,z}) + \sum_{n\geq0}\Delta L_M(Z_{\tau_n}) \\ &\leq\; L_M(Z_T^{0,t,z}) - \varepsilon N_T(\alpha), \end{aligned}$$

where we recall that $N_T(\alpha)$ is the number of trading times over the whole horizon T. This shows that

$$N_T(\alpha) \quad \leq \quad \frac{1}{\varepsilon} L_M(Z_T^{0,t,z}) \; < \; \infty \quad a.s.$$

In other words, we see that, under the presence of fixed transaction fee, the number of intervention times over a finite interval for an admissible trading strategy is finite almost surely.

The dynamic programming equation associated to the control problem (5.5.2) is

$$\begin{aligned} \min\Big[-\frac{\partial w}{\partial t} - \frac{\partial w}{\partial \theta} - \mathcal{L}w \; , \; w - \mathcal{H}_\varepsilon w \Big] &= 0, \quad \text{in } [0,T) \times \bar{\mathcal{S}}_\varepsilon, & (5.5.5) \\ \min\left[w - U_{L_\varepsilon} \; , \; w - \mathcal{H}_\varepsilon w \right] &= 0, \quad \text{in } \{T\} \times \bar{\mathcal{S}}_\varepsilon. & (5.5.6) \end{aligned}$$

The main result of this section is stated as follows.

Theorem 5.5.1 *(1) The sequence $(v_\varepsilon)_\varepsilon$ is nonincreasing, and converges pointwise on $[0,T] \times (\bar{\mathcal{S}} \setminus \partial_L \mathcal{S})$ towards v as ε goes to zero.*
(2) For any $\varepsilon > 0$, the value function v_ε is continuous on $[0,T) \times \mathcal{S}_\varepsilon$, and is the unique (in $[0,T) \times \mathcal{S}_\varepsilon$) constrained viscosity solution to (5.5.5)-(5.5.6), satisfying the growth condition:

$$|v_\varepsilon(t,z,\theta)| \quad \leq \quad K(1 + L_M(z)^\gamma), \quad \forall (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}_\varepsilon, \tag{5.5.7}$$

for some positive constant K, and the boundary condition:

$$\begin{aligned} \lim_{(t',z',\theta') \to (t,z,\theta)} v_\varepsilon(t',z',\theta') &= v(t,z,\theta) \\ &= U(0), \quad \forall (t, z = (0,0,p), \theta) \in [0,T] \times D_0. \end{aligned} \tag{5.5.8}$$

We first prove the convergence of the sequence of value functions (v_ε).

Proof of Theorem 5.5.1 (1).
Notice that for any $0 < \varepsilon_1 \leq \varepsilon_2$, we have $L_{\varepsilon_2} \leq L_{\varepsilon_1} \leq L$, $\mathcal{A}^{\varepsilon_2}(t,z,\theta) \subset \mathcal{A}^{\varepsilon_1}(t,z,\theta) \subset \mathcal{A}(t,z,\theta)$, for $t \in [0,T]$, $(z,\theta) \in \bar{\mathcal{S}}_{\varepsilon_2} \subset \bar{\mathcal{S}}_{\varepsilon_1} \subset \bar{\mathcal{S}}$, and for $\alpha \in \mathcal{A}^{\varepsilon_2}(t,z,\theta)$, $L_{\varepsilon_2}(Z^{\varepsilon_2}, \Theta) \leq L_{\varepsilon_2}(Z^{\varepsilon_1}, \Theta) \leq L_{\varepsilon_1}(Z^{\varepsilon_1}, \Theta) \leq L(Z, \Theta)$. This shows that the sequence (v_ε) is nonincreasing, and is upper-bounded by the value function v without transaction fee, so that

$$\lim_{\varepsilon \downarrow 0} v_\varepsilon(t,z,\theta) \quad \leq \quad v(t,z,\theta), \quad \forall (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}. \tag{5.5.9}$$

Fix now some point $(t,z,\theta) \in [0,T] \times (\bar{\mathcal{S}} \setminus \partial_L \mathcal{S})$. From the representation (5.3.13) of $v(t,z,\theta)$, there exists for any $n \geq 1$, an $1/n$-optimal control $\alpha^{(n)} = (\tau_k^{(n)}, \zeta_k^{(n)})_k \in \mathcal{A}^b_{\ell_+}(t,z,\theta)$ with associated state process $(Z^{(n)} = (X^{(n)}, Y^{(n)}, P), \Theta^{(n)})$ and number of trading times $N^{(n)}$:

$$\mathbb{E}[U(X_T^{(n)})] \quad \geq \quad v(t,z,\theta) - \frac{1}{n}. \tag{5.5.10}$$

We denote by $(Z^{\varepsilon,(n)}, \Theta^{(n)}) = (X^{\varepsilon,(n)}, Y^{(n)}, P), \Theta^{(n)})$ the state process controlled by $\alpha^{(n)}$ in the model with transaction fee ε (only the cash component is affected by ε), and we observe that for all $t \leq s \leq T$,

$$X_s^{\varepsilon,(n)} \;=\; X_s^{(n)} - \varepsilon N_s^{(n)} \nearrow X_s^{(n)}, \quad \text{as } \varepsilon \text{ goes to zero.} \tag{5.5.11}$$

Given n, we consider the family of stopping times:

$$\sigma_\varepsilon^{(n)} \;=\; \inf\{s \geq t:\ L(Z_s^{\varepsilon,(n)}, \Theta_s^{(n)}) \leq \varepsilon\} \wedge T, \qquad \varepsilon > 0.$$

Let us prove that

$$\lim_{\varepsilon \searrow 0} \sigma_\varepsilon^{(n)} \;=\; T \;\; a.s. \tag{5.5.12}$$

Observe that for $0 < \varepsilon_1 \leq \varepsilon_2$, $X_s^{\varepsilon_2,(n)} \leq X_s^{\varepsilon_1,(n)}$, and so $L(Z_s^{\varepsilon_2,(n)}, \Theta_s) \leq L(Z_s^{\varepsilon_1,(n)}, \Theta_s)$ for $t \leq s \leq T$. This implies clearly that the sequence $(\sigma_\varepsilon^{(n)})_\varepsilon$ is nonincreasing. Since this sequence is bounded by T, it admits a limit, denoted by $\sigma_0^{(n)} = \lim_{\varepsilon \downarrow 0} \uparrow \sigma_\varepsilon^{(n)}$. Now, by definition of $\sigma_\varepsilon^{(n)}$, we have $L(Z^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)}}, \Theta^{(n)}_{\sigma_\varepsilon^{(n)}}) \leq \varepsilon$, for all $\varepsilon > 0$. By sending ε to zero, we then get with (5.5.11):

$$L(Z^{(n)}_{\sigma_0^{(n)},-}, \Theta^{(n)}_{\sigma_0^{(n)},-}) \;=\; 0 \;\; a.s.$$

Recalling the definition of $\mathcal{A}^b_{\ell_+}(t,z,\theta)$, this implies that $\sigma_0^{(n)} = \tau_k^{(n)}$ for some $k \in \{1, \ldots, N^{(n)} + 1\}$ with the convention $\tau^{(n)}_{N^{(n)}+1} = T$. If $k \leq N^{(n)}$, arguing as in (5.3.15), we get a contradiction with the solvency constraints. Hence we get $\sigma_0^{(n)} = T$.

Consider now the trading strategy $\tilde{\alpha}^{\varepsilon,(n)} \in \mathcal{A}$ consisting in following $\alpha^{(n)}$ until time $\sigma_\varepsilon^{(n)}$ and liquidating all the stock shares at time $\sigma_\varepsilon^{(n)}$, i.e.

$$\tilde{\alpha}^{\varepsilon,(n)} \;=\; (\tau_k^{(n)}, \zeta_k^{(n)})\mathbf{1}_{\tau_k < \sigma_\varepsilon^{(n)}} \cup (\sigma_\varepsilon^{(n)}, -Y_{\sigma_\varepsilon^{(n)},-}).$$

We denote by $(\tilde{Z}^{\varepsilon,(n)} = (\tilde{X}^{\varepsilon,(n)}, \tilde{Y}^{\varepsilon,(n)}, P), \tilde{\Theta}^{\varepsilon,(n)})$ the associated state process in the market with transaction fee ε. By construction, we have for all $t \leq s < \sigma_\varepsilon^{(n)}$: $L(\tilde{Z}_s^{\varepsilon,(n)}, \tilde{\Theta}_s^{\varepsilon,(n)}) = L(Z_s^{\varepsilon,(n)}, \Theta_s^{(n)}) \geq \varepsilon$, and thus $L_\varepsilon(\tilde{Z}_s^{\varepsilon,(n)}, \tilde{\Theta}_s^{\varepsilon,(n)}) \geq 0$. At the transaction time $\sigma_\varepsilon^{(n)}$, we then have $\tilde{X}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)}} = L(\tilde{Z}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)},-}, \tilde{\Theta}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)},-}) - \varepsilon = L(Z^{(n)}_{\sigma_\varepsilon^{\varepsilon,(n)},-}, \Theta^{(n)}_{\sigma_\varepsilon^{(n)},-}) - \varepsilon$, $\tilde{Y}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)}} = 0$. After time $\sigma_\varepsilon^{(n)}$, there is no more transaction in $\tilde{\alpha}^{\varepsilon,(n)}$, and so

$$\tilde{X}_s^{\varepsilon,(n)} = \tilde{X}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)}} \;=\; L(Z^{(n)}_{\sigma_\varepsilon^{\varepsilon,(n)},-}, \Theta^{(n)}_{\sigma_\varepsilon^{(n)},-}) - \varepsilon \;\geq\; 0, \tag{5.5.13}$$

$$\tilde{Y}_s^{\varepsilon,(n)} = \tilde{Y}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)}} \;=\; 0, \qquad \sigma_\varepsilon^{(n)} \leq s \leq T, \tag{5.5.14}$$

and thus $L_\varepsilon(\tilde{Z}_s^{\varepsilon,(n)}, \tilde{\Theta}_s^{\varepsilon,(n)}) = \tilde{X}_s^{\varepsilon,(n)} \geq 0$ for $\sigma_\varepsilon^{(n)} \leq s \leq T$. This shows that $\tilde{\alpha}^{\varepsilon,(n)}$ lies in $\mathcal{A}^\varepsilon(t,z,\theta)$, and thus by definition of v_ε:

$$v_\varepsilon(t,z) \;\geq\; \mathbb{E}[U_{L_\varepsilon}(\tilde{Z}_T^{\varepsilon,(n)}, \tilde{\Theta}_T^{\varepsilon,(n)})]. \tag{5.5.15}$$

Let us check that given n,

$$\lim_{\varepsilon\downarrow 0} L_\varepsilon(\tilde{Z}_T^{\varepsilon,(n)}, \tilde{\Theta}_T^{\varepsilon,(n)}) \;=\; X_T^{(n)}, \qquad a.s. \tag{5.5.16}$$

To alleviate notations, we set $N = N_T^{(n)}$ the total number of trading times of $\alpha^{(n)}$. If the last trading time of $\alpha^{(n)}$ occurs strictly before T, then we do not trade anymore until the final horizon T, and so

$$X_T^{(n)} \;=\; X_{\tau_N}^{(n)}, \quad \text{and} \quad Y_T^{(n)} \;=\; Y_{\tau_N}^{(n)} \;=\; 0, \quad \text{on } \{\tau_N < T\}. \tag{5.5.17}$$

By (5.5.12), we have for ε small enough: $\sigma_\varepsilon^{(n)} > \tau_N$, and so $\tilde{X}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)},-} = X_{\tau_N}^{\varepsilon,(n)}$, $\tilde{Y}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)},-} = Y_{\tau_N}^{(n)}$ $= 0$. The final liquidation at time $\sigma_\varepsilon^{(n)}$ yields: $\tilde{X}_T^{\varepsilon,(n)} = \tilde{X}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)}} = \tilde{X}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)},-} - \varepsilon = X_{\tau_N}^{\varepsilon,(n)} - \varepsilon$, and $\tilde{Y}_T^{\varepsilon,(n)} = \tilde{Y}^{\varepsilon,(n)}_{\sigma_\varepsilon^{(n)}} = 0$. We then obtain

$$\begin{aligned} L_\varepsilon(\tilde{Z}_T^{\varepsilon,(n)}, \tilde{\Theta}_T^{\varepsilon,(n)}) &= \max\Big(\tilde{X}_T^{\varepsilon,(n)}, L(\tilde{Z}_T^{\varepsilon,(n)}, \tilde{\Theta}_T^{\varepsilon,(n)}) - \varepsilon\Big) \\ &= \tilde{X}_T^{\varepsilon,(n)} \;=\; X_{\tau_N}^{\varepsilon,(n)} - \varepsilon \qquad \text{on } \{\tau_N < T\} \\ &= X_T^{(n)} - (1+N)\varepsilon \qquad \text{on } \{\tau_N < T\}, \end{aligned}$$

by (5.5.11) and (5.5.17), which shows that the convergence in (5.5.16) holds on $\{\tau_N < T\}$. If the last trading of $\alpha^{(n)}$ occurs at time T, this means that we liquidate all stock shares at T, and so

$$X_T^{(n)} \;=\; L(Z_{T-}^{(n)}, \Theta_{T-}^{(n)}), \;\; Y_T^{(n)} \;=\; 0 \qquad \text{on } \{\tau_N = T\}. \tag{5.5.18}$$

On the other hand, by (5.5.13)-(5.5.14), we have

$$\begin{aligned} L_\varepsilon(\tilde{Z}_T^{\varepsilon,(n)}, \tilde{\Theta}_T^{\varepsilon,(n)}) \;=\; \tilde{X}_T^{\varepsilon,(n)} &= L(Z^{(n)}_{\sigma_\varepsilon^{\varepsilon,(n)},-}, \Theta^{(n)}_{\sigma_\varepsilon^{(n)},-}) - \varepsilon \\ &\longrightarrow L(Z_{T-}^{(n)}, \Theta_{T-}^{(n)}) \qquad \text{as } \varepsilon \text{ goes to zero,} \end{aligned}$$

by (5.5.12). Together with (5.5.18), this implies that the convergence in (5.5.16) also holds on $\{\tau_N = T\}$, and thus almost surely. Since $0 \le L_\varepsilon \le L$, we immediately see by Proposition 5.3.1 that the sequence $\{U_{L_\varepsilon}(\tilde{Z}_T^{\varepsilon,(n)}, \tilde{\Theta}_T^{\varepsilon,(n)}), \varepsilon > 0\}$ is uniformly integrable, so that by sending ε to zero in (5.5.15) and using (5.5.16), we get

$$\lim_{\varepsilon\downarrow 0} v_\varepsilon(t,z,\theta) \;\ge\; \mathbb{E}[U(X_T^{(n)})] \;\ge\; v(t,z) - \frac{1}{n},$$

from (5.5.10). By sending n to infinity, and recalling (5.5.9), this completes the proof of assertion (1) in Theorem 5.5.1. □

We now turn to the viscosity characterization of v_ε. The viscosity property of v_ε is proved similarly as for v, and is then omitted. From Proposition 5.3.1, and since $0 \le v_\varepsilon$

$\leq v$, we know that the value functions v_ε lie in the set of functions satisfying the growth condition in (5.5.7), i.e.

$$\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}}_\varepsilon) \quad = \quad \Big\{ w : [0,T] \times \bar{\mathcal{S}}_\varepsilon \to \mathbb{R}, \ \sup_{[0,T]\times\bar{\mathcal{S}}_\varepsilon} \frac{|w(t,z,\theta)|}{1 + L_M(z)^\gamma} \ < \ \infty \Big\}.$$

The boundary property (5.5.8) is immediate. Indeed, fix $(t, z = (x,0,p), \theta) \in [0,T] \times \partial_y \mathcal{S}_\varepsilon$, and consider an arbitrary sequence $(t_n, z_n = (x_n, y_n, p_n), \theta_n)_n$ in $[0,T] \times \bar{\mathcal{S}}_\varepsilon$ converging to (t,z,θ). Since $0 \leq L_\varepsilon(z_n, \theta_n) = \max(x_n, L(z_n,\theta_n) - \varepsilon)$, and y_n goes to zero, this implies that for n large enough, $x_n = L_\varepsilon(z_n,\theta_n) \geq 0$. By considering from (t_n, z_n, θ_n) the admissible strategy of doing none transaction, which leads to a final liquidation value $X_T = x_n$, we have $U(x_n) \leq v_\varepsilon(t_n, z_n, \theta_n) \leq v(t_n, z_n, \theta_n)$. Recalling Corollary 5.3.1, we then obtain the continuity of v_ε on $\partial_y \mathcal{S}_\varepsilon$ with $v_\varepsilon(t,z,\theta) = U(x) = v(t,z,\theta)$ for $(z,\theta) = (x,0,p,\theta) \in \partial_y \mathcal{S}_\varepsilon$, and in particular (5.5.8). Finally, we address the uniqueness issue, which is a direct consequence of the following comparison principle for constrained (discontinuous) viscosity solution to (5.5.5)-(5.5.6).

Theorem 5.5.2 *(Comparison principle)*
Suppose $u \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}}_\varepsilon)$ is a usc viscosity subsolution to (5.5.5)-(5.5.6) *on* $[0,T] \times \bar{\mathcal{S}}_\varepsilon$, *and $w \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}}_\varepsilon)$ is a lsc viscosity supersolution to* (5.5.5)-(5.5.6) *on* $[0,T] \times \mathcal{S}_\varepsilon$ *such that*

$$u(t,z,\theta) \quad \leq \quad \liminf_{\substack{(t',z',\theta') \to (t,z,\theta) \\ (t',z',\theta') \in [0,T) \times \mathcal{S}_\varepsilon}} \quad w(t',z',\theta'), \quad \forall (t,z,\theta) \in [0,T] \times D_0. \qquad (5.5.19)$$

Then,

$$u \quad \leq \quad w \quad \text{ on } \quad [0,T] \times \mathcal{S}_\varepsilon. \qquad (5.5.20)$$

Notice that with respect to usual comparison principles for parabolic PDEs where we compare a viscosity subsolution and a viscosity supersolution from the inequalities on the domain and at the terminal date, we require here in addition a comparison on the boundary D_0 due to the non smoothness of the domain $\bar{\mathcal{S}}_\varepsilon$ on this right angle of the boundary. A similar feature appears also in [53], and we shall only emphasize the main arguments adapted from [4], for proving the comparison principle.

Proof of Theorem 5.5.2.

Let u and w as in Theorem 5.5.2, and (re)define w on $[0,T] \times \partial\mathcal{S}_\varepsilon$ by

$$w(t,z,\theta) \quad = \quad \liminf_{\substack{(t',z',\theta') \to (t,z,\theta) \\ (t',z',\theta') \in [0,T) \times \mathcal{S}_\varepsilon}} \quad w(t',z',\theta'), \qquad (t,z,\theta) \in [0,T] \times \partial\mathcal{S}_\varepsilon. \qquad (5.5.21)$$

In order to obtain the comparison result (5.5.20), it suffices to prove that $\sup_{[0,T]\times\bar{\mathcal{S}}_\varepsilon}(u-w) \leq 0$, and we shall argue by contradiction by assuming that

$$\sup_{[0,T]\times\bar{\mathcal{S}}_\varepsilon} (u - w) \quad > \quad 0. \qquad (5.5.22)$$

• *Step 1. Construction of a strict viscosity supersolution.*

Consider the function defined on $[0,T] \times \bar{\mathcal{S}}_\varepsilon$ by

$$\psi(t,z,\theta) \;=\; e^{\rho'(T-t)} L_M(z)^{\gamma'}, \qquad t \in [0,T], (z,\theta) = (x,y,p,\theta) \in \bar{\mathcal{S}}_\varepsilon,$$

where $\rho' > 0$, and $\gamma' \in (0,1)$ will be chosen later. The function ψ is smooth C^2 on $[0,T) \times (\bar{\mathcal{S}}_\varepsilon \setminus D_0)$, and by the same calculations as in (5.3.10), we see that by choosing $\rho' > \frac{\gamma'}{1-\gamma'} \frac{b^2}{2\sigma^2}$, then

$$-\frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \theta} - \mathcal{L}\psi \;>\; 0 \qquad \text{on } [0,T) \times (\bar{\mathcal{S}}_\varepsilon \setminus D_0). \tag{5.5.23}$$

Moreover, from (5.5.4), we have

$$\begin{aligned}(\psi - \mathcal{H}_\varepsilon \psi)(t,z,\theta) \;&=\; e^{\rho'(T-t)}\Big[L_M(z)^{\gamma'} - (\mathcal{H}_\varepsilon L_M(z))^{\gamma'}\Big] \;=:\; \Delta(t,z) \\ &>\; 0 \qquad \text{on } [0,T] \times \bar{\mathcal{S}}_\varepsilon.\end{aligned} \tag{5.5.24}$$

For $m \geq 1$, we denote by

$$\tilde{u}(t,z,\theta) \;=\; e^t u(t,z,\theta), \quad \text{and} \quad \tilde{w}_m(t,z,\theta) \;=\; e^t[w(t,z,\theta) + \frac{1}{m}\psi(t,z,\theta)].$$

From the viscosity subsolution property of u, we immediately see that $\tilde{u}$ is a viscosity subsolution to

$$\min\,[\tilde{u} - \frac{\partial \tilde{u}}{\partial t} - \frac{\partial \tilde{u}}{\partial \theta} - \mathcal{L}\tilde{u}\;,\; \tilde{u} - \mathcal{H}_\varepsilon \tilde{u}] \;\leq\; 0, \quad \text{on } [0,T) \times \bar{\mathcal{S}}_\varepsilon \tag{5.5.25}$$

$$\min\,[\tilde{u} - \tilde{U}_{L_\varepsilon},\; \tilde{u} - \mathcal{H}_\varepsilon \tilde{u}] \;\leq\; 0, \quad \text{on } \{T\} \times \bar{\mathcal{S}}_\varepsilon, \tag{5.5.26}$$

where we set $\tilde{U}_{L_\varepsilon}(z,\theta) = e^T U_{L_\varepsilon}(z,\theta)$. From the viscosity supersolution property of w, and the relations (5.5.23)-(5.5.24), we also derive that $\tilde{w}_m$ is a viscosity supersolution to

$$\tilde{w}_m - \frac{\partial \tilde{w}_m}{\partial t} - \frac{\partial \tilde{w}_m}{\partial \theta} - \mathcal{L}\tilde{w}_m \;\geq\; 0 \qquad \text{on } [0,T) \times (\mathcal{S}_\varepsilon \setminus D_0) \tag{5.5.27}$$

$$\tilde{w}_m - \mathcal{H}_\varepsilon \tilde{w}_m \;\geq\; \frac{1}{m}\Delta \quad \text{on } [0,T] \times \mathcal{S}_\varepsilon. \tag{5.5.28}$$

$$\tilde{w}_m - \tilde{U}_{L_\varepsilon} \;\geq\; 0 \qquad \text{on } \{T\} \times \mathcal{S}_\varepsilon. \tag{5.5.29}$$

On the other hand, from the growth condition on u and w in $\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}}_\varepsilon)$, and by choosing $\gamma' \in (\gamma, 1)$, we have for all $(t,\theta) \in [0,T]^2$,

$$\lim_{|z| \to \infty} (u - w_m)(t,z,\theta) \;=\; -\infty.$$

Therefore, the usc function $\tilde{u} - \tilde{w}_m$ attains its supremum on $[0,T] \times \bar{\mathcal{S}}_\varepsilon$, and from (5.5.22), there exists m large enough, and $(\bar{t},\bar{z},\bar{\theta}) \in [0,T] \times \bar{\mathcal{S}}_\varepsilon$ s.t.

$$\tilde{M} \;=\; \sup_{[0,T] \times \bar{\mathcal{S}}_\varepsilon} (\tilde{u} - \tilde{w}_m) \;=\; (\tilde{u} - \tilde{w}_m)(\bar{t},\bar{z},\bar{\theta}) \;>\; 0. \tag{5.5.30}$$

• *Step 2.* From the boundary condition (5.5.19), we know that $(\bar{z}, \bar{\theta})$ cannot lie in D_0, and we have then two possible cases:

(i) $(\bar{z}, \bar{\theta}) \in \mathcal{S}_\varepsilon \setminus D_0$

(ii) $(\bar{z}, \bar{\theta}) \in \partial\mathcal{S}_\varepsilon \setminus D_0$.

The case (i) where $(\bar{z}, \bar{\theta})$ lies in $\mathcal{S}_\varepsilon$ is standard in the comparison principle for (nonconstained) viscosity solutions, and we focus here on the case (ii), which is specific to constrained viscosity solutions. From (5.5.21), there exists a sequence $(t_n, z_n, \theta_n)_{n\geq 1}$ in $[0, T) \times \mathcal{S}_\varepsilon$ such that

$$(t_n, z_n, \theta_n, \tilde{w}_m(t_n, z_n, \theta_n)) \longrightarrow (\bar{t}, \bar{z}, \bar{\theta}, \tilde{w}_m(\bar{t}, \bar{z}, \bar{\theta})) \qquad \text{as } n \to \infty.$$

We then set $\delta_n = |z_n - \bar{z}| + |\theta_n - \bar{\theta}|$, and consider the function Φ_n defined on $[0, T] \times (\bar{\mathcal{S}}_\varepsilon)^2$ by:

$$\begin{aligned}
\Phi_n(t, z, \theta, z', \theta') &= \tilde{u}(t, z, \theta) - \tilde{w}_m(t, z', \theta') - \varphi_n(t, z, \theta, z', \theta') \\
\varphi_n(t, z, \theta, z', \theta') &= |t - \bar{t}|^2 + |z - \bar{z}|^4 + |\theta - \bar{\theta}|^4 \\
&\quad + \frac{|z - z'|^2 + |\theta - \theta'|^2}{2\delta_n} + \Big(\frac{d(z', \theta')}{d(z_n, \theta_n)} - 1\Big)^4.
\end{aligned}$$

Here, $d(z, \theta)$ denotes the distance from (z, θ) to $\partial\mathcal{S}_\varepsilon$. Since $(\bar{z}, \bar{\theta}) \notin D_0$, there exists an open neighborhood $\bar{\mathcal{V}}$ of $(\bar{z}, \bar{\theta})$ satisfying $\bar{\mathcal{V}} \cap D_0 = \emptyset$, such that the function $d(.)$ is twice continuously differentiable with bounded derivatives. This is well known (see e.g. [36]) when $(\bar{z}, \bar{\theta})$ lies in the smooth parts of the boundary $\partial\mathcal{S}_\varepsilon \setminus (D_{1,\varepsilon} \cup D_{2,\varepsilon})$. This is also true for $(\bar{z}, \bar{\theta}) \in D_{k,\varepsilon}$ for $k \in \{1, 2\}$. Indeed, at these corner lines, the inner normal vectors form an acute angle (positive scalar product), and thus one can extend from $(\bar{z}, \bar{\theta})$ the boundary to a smooth boundary so that the distance d is equal, locally on the neighborhood, to the distance to this smooth boundary. From the growth conditions on u and w in $\mathcal{G}_\gamma([0, T] \times \bar{\mathcal{S}}_\varepsilon)$, there exists a sequence $(\hat{t}_n, \hat{z}_n, \hat{\theta}_n, \hat{z}'_n, \hat{\theta}'_n)$ attaining the maximum of the usc Φ_n on $[0, T] \times (\bar{\mathcal{S}}_\varepsilon)^2$. By standard arguments (see e.g. [4] or [53]), we have

$$(\hat{t}_n, \hat{z}_n, \hat{\theta}_n, \hat{z}'_n, \hat{\theta}'_n) \longrightarrow (\bar{t}, \bar{z}, \bar{\theta}, \bar{z}, \bar{\theta}) \tag{5.5.31}$$

$$\frac{|\hat{z}_n - \hat{z}'_n|^2 + |\hat{\theta}_n - \hat{\theta}'_n|^2}{2\delta_n} + \Big(\frac{d(\hat{z}'_n, \hat{\theta}'_n)}{d(z_n, \theta_n)} - 1\Big)^4 \longrightarrow 0 \tag{5.5.32}$$

$$\tilde{u}(\hat{t}_n, \hat{z}_n, \hat{\theta}_n) - \tilde{w}_m(\hat{t}_n, \hat{z}'_n, \hat{\theta}'_n) \longrightarrow (\tilde{u} - \tilde{w}_m)(\bar{t}, \bar{z}, \bar{\theta}). \tag{5.5.33}$$

The convergence in (5.5.32) shows in particular that for n large enough, $d(\hat{z}'_n, \hat{\theta}'_n) \geq d(z_n, \theta_n)/2$ > 0, and so $(\hat{z}'_n, \hat{\theta}'_n) \in \mathcal{S}_\varepsilon$. From the convergence in (5.5.31), we may also assume that for n large enough, $(\hat{z}_n, \hat{\theta}_n)$, $(\hat{z}'_n, \hat{\theta}'_n)$ lie in the neighborhood $\bar{\mathcal{V}}$ of $(\bar{z}, \bar{\theta})$ so that the derivatives upon order 2 of $d(.)$ at $(\hat{z}_n, \hat{\theta}_n)$ and $(\hat{z}'_n, \hat{\theta}'_n)$ exist and are bounded.

• *Step 3.* We show that for n large enough,

$$\tilde{u}(\hat{t}_n, \hat{z}_n, \hat{\theta}_n) - \mathcal{H}_\varepsilon\tilde{u}(\hat{t}_n, \hat{z}_n) > 0. \tag{5.5.34}$$

Otherwise, up to a subsequence, we would have for all n:

$$\tilde{u}(\hat{t}_n, \hat{z}_n, \hat{\theta}_n) - \mathcal{H}_\varepsilon \tilde{u}(\hat{t}_n, \hat{z}_n) \quad \leq \quad 0.$$

By sending n to infinity, and from the upper-semicontinuity of $\mathcal{H}_\varepsilon \tilde{u}$, we get with (5.5.31): $-\infty < \tilde{u}(\bar{t}, \bar{z}, \bar{\theta}) \leq \mathcal{H}_\varepsilon \tilde{u}(\bar{t}, \bar{z}, \bar{\theta})$, which shows in particular that $\mathcal{C}_\varepsilon(\bar{z}, \bar{\theta})$ is not empty. Moreover, by the viscosity supersolution property (5.5.28), we have

$$\tilde{w}_m(\hat{t}_n, \hat{z}'_n, \hat{\theta}'_n) - \mathcal{H}_\varepsilon \tilde{w}_m(\hat{t}_n, \hat{z}'_n, \hat{\theta}'_n) \quad \geq \quad \frac{1}{m}\Delta(\hat{t}_n, \hat{z}'_n, \hat{\theta}'_n).$$

By substracting the two previous inequalities, we would get

$$\tilde{u}(\hat{t}_n, \hat{z}_n, \hat{\theta}_n) - \tilde{w}_m(\hat{t}_n, \hat{z}'_n, \hat{\theta}'_n) \quad \leq \quad \mathcal{H}_\varepsilon \tilde{u}(\hat{t}_n, \hat{z}_n) - \mathcal{H}_\varepsilon \tilde{w}_m(\hat{t}_n, \hat{z}'_n, \hat{\theta}'_n) - \frac{1}{m}\Delta(\hat{t}_n, \hat{z}'_n, \hat{\theta}'_n).$$

By sending n to infinity, and from the upper-semicontinuity of $\mathcal{H}_\varepsilon \tilde{u}$, the lower-semicontinuity of $\mathcal{H}_\varepsilon \tilde{w}_m$ and Δ, this yields with (5.5.31), (5.5.33)

$$(\tilde{u} - \tilde{w}_m)(\bar{t}, \bar{z}, \bar{\theta}) \quad \leq \quad \mathcal{H}_\varepsilon \tilde{u}(\bar{t}, \bar{z}, \bar{\theta}) - \mathcal{H}_\varepsilon \tilde{w}_m(\bar{t}, \bar{z}, \bar{\theta}) - \frac{1}{m}\Delta(\bar{t}, \bar{z}, \bar{\theta}).$$

Now, by compactness of $\mathcal{C}_\varepsilon(\bar{z}, \bar{\theta}) \neq \emptyset$, there exists $\bar{e} \in \mathcal{C}_\varepsilon(\bar{z}, \bar{\theta})$ such that $\mathcal{H}_\varepsilon \tilde{u}(\bar{t}, \bar{z}, \bar{\theta}) = \tilde{u}(t, \Gamma_\varepsilon(\bar{z}, \bar{\theta}, \bar{e}), 0)$ and so

$$\begin{aligned} \tilde{M} \;=\; (\tilde{u} - \tilde{w}_m)(\bar{t}, \bar{z}, \bar{\theta}) \quad &\leq \quad \tilde{u}(\bar{t}, \Gamma_\varepsilon(\bar{z}, \bar{\theta}, \bar{e}), 0) - \tilde{w}_m(\bar{t}, \Gamma_\varepsilon(\bar{z}, \bar{\theta}, \bar{e}), 0) - \frac{1}{m}\Delta(\bar{t}, \bar{z}, \bar{\theta}) \\ &\leq \quad \tilde{M} - \frac{1}{m}\Delta(\bar{t}, \bar{z}, \bar{\theta}), \end{aligned}$$

a contradiction.

• *Step 4.* We check that, up to a subsequence, $\hat{t}_n < T$ for all n. On the contrary, $\hat{t}_n = \bar{t} = T$ for n large enough, and we would get from (5.5.34) and the viscosity subsolution property (5.5.26):

$$\tilde{u}(T, \hat{z}_n, \hat{\theta}_n) \quad \leq \quad \tilde{U}_{L_\varepsilon}(\hat{z}_n, \hat{\theta}_n).$$

Moreover, by (5.5.29), we have $\tilde{w}_m(T, \hat{z}'_n, \hat{\theta}'_n) \geq \tilde{U}_{L_\varepsilon}(\hat{z}'_n, \hat{\theta}'_n)$, which combined with the former inequality, implies

$$\tilde{u}(T, \hat{z}_n, \hat{\theta}_n) - \tilde{w}_m(T, \hat{z}'_n, \hat{\theta}'_n) \quad \leq \quad \tilde{U}_{L_\varepsilon}(\hat{z}_n, \hat{\theta}_n) - \tilde{U}_{L_\varepsilon}(\hat{z}'_n, \hat{\theta}'_n).$$

By sending n to infinity, this yields with (5.5.31), (5.5.33) and continuity of $\tilde{U}_{L_\varepsilon}$: $\tilde{M} = (\tilde{u} - \tilde{w}_m)(\bar{t}, \bar{z}, \bar{\theta}) \leq 0$, a contradiction with (5.5.30).

• *Step 5.* We use the viscosity subsolution property (5.5.25) of $\tilde{u}$ at $(\hat{t}_n, \hat{z}_n, \hat{\theta}_n) \in [0, T) \times \bar{\mathcal{S}}_\varepsilon$, which is written by (5.5.34) as

$$(\tilde{u} - \frac{\partial \tilde{u}}{\partial t} - \frac{\partial \tilde{u}}{\partial \theta} - \mathcal{L}\tilde{u})(\hat{t}_n, \hat{z}_n, \hat{\theta}_n) \quad \leq \quad 0. \tag{5.5.35}$$

The above inequality is understood in the viscosity sense, and applied with the test function $(t,z,\theta) \to \varphi_n(t,z,\theta,\hat{z}'_n,\hat{\theta}'_n)$, which is C^2 in the neighborhood $[0,T]\times\bar{\mathcal{V}}$ of $(\hat{t}_n,\hat{z}_n,\hat{\theta}_n)$. We also write the viscosity supersolution property (5.5.27) of $\tilde{w}_m$ at $(\hat{t}_n,\hat{z}'_n,\hat{\theta}'_n) \in [0,T) \times (\mathcal{S}_\varepsilon \backslash D_0)$:

$$(\tilde{w}_m - \frac{\partial \tilde{w}_m}{\partial t} - \frac{\partial \tilde{w}_m}{\partial \theta} - \mathcal{L}\tilde{w}_m)(\hat{t}_n,\hat{z}'_n,\hat{\theta}'_n) \quad \geq \quad 0. \tag{5.5.36}$$

The above inequality is again understood in the viscosity sense, and applied with the test function $(t,z',\theta') \to -\varphi_n(t,\hat{z}_n,\hat{\theta}_n,z',\theta')$, which is C^2 in the neighborhood $[0,T] \times \bar{\mathcal{V}}$ of $(\hat{t}_n,\hat{z}'_n,\hat{\theta}'_n)$. The conclusion is achieved by arguments similar to [53]: we invoke Ishii's Lemma, substract the two inequalities (5.5.35)-(5.5.36), and finally get the required contradiction $\tilde{M} \leq 0$ by sending n to infinity with (5.5.31)-(5.5.32)-(5.5.33). □

5.6 An approximating problem with utility penalization

We consider in this section another perturbation of our initial optimization problem by adding a cost ε to the utility at each trading. We then define the value function $\bar{v}_\varepsilon$ on $[0,T] \times \bar{\mathcal{S}}$ by

$$\bar{v}_\varepsilon(t,z,\theta) \quad = \quad \sup_{\alpha \in \mathcal{A}^b_\ell(t,z,\theta)} \mathbb{E}\Big[U_L(Z_T,\Theta_T) - \varepsilon N_T(\alpha)\Big], \quad (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}. \tag{5.6.1}$$

The convergence of this approximation is immediate.

Proposition 5.6.1 *The sequence $(\bar{v}_\varepsilon)_\varepsilon$ is nondecreasing and converges pointwise on $[0,T] \times \bar{\mathcal{S}}$ towards v as ε goes to zero.*

Proof. It is clear that the sequence $(\bar{v}_\varepsilon)_\varepsilon$ is nondecreasing and that $\bar{v}_\varepsilon \leq v$ on $[0,T] \times \bar{\mathcal{S}}$ for any $\varepsilon > 0$. Let us prove that $\lim_{\varepsilon \searrow 0} \bar{v}_\varepsilon = v$. Fix $n \in \mathbb{N}^*$ and $(t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}$ and consider some $\alpha^{(n)} \in \mathcal{A}^b_\ell(t,z,\theta)$ such that

$$\mathbb{E}\Big[U_L(Z^{(n)}_T,\Theta^{(n)}_T)\Big] \quad \geq \quad v(t,z,\theta) - \frac{1}{n},$$

where $(Z^{(n)},\Theta^{(n)})$ is the associated controlled process. From the monotone convergence theorem, we then get

$$\lim_{\varepsilon \searrow 0} \bar{v}_\varepsilon(t,z,\theta) \quad \geq \quad \mathbb{E}\Big[U_L\Big(Z^{(n)}_T,\Theta^{(n)}_T\Big)\Big] \;\geq\; v(t,z,\theta) - \frac{1}{n}.$$

By the arbitrariness of $n \in \mathbb{N}^*$, we conclude that $\lim_\varepsilon \bar{v}_\varepsilon \geq v$, which ends the proof since we already have $\bar{v}_\varepsilon \leq v$. □

The nonlocal impulse operator $\bar{\mathcal{H}}^\varepsilon$ associated to (5.6.1) is given by

$$\bar{\mathcal{H}}_\varepsilon \varphi(t,z,\theta) \quad = \quad \mathcal{H}\varphi(t,z,\theta) - \varepsilon,$$

and we consider the corresponding dynamic programming equation:

$$\begin{aligned} \min\Big[-\frac{\partial w}{\partial t} - \frac{\partial w}{\partial \theta} - \mathcal{L}w \ , \ w - \bar{\mathcal{H}}_\varepsilon w \Big] &= 0, \quad \text{in } [0,T) \times \bar{\mathcal{S}}, && (5.6.2)\\ \min\big[w - U_L \ , \ w - \bar{\mathcal{H}}_\varepsilon w \big] &= 0, \quad \text{in } \{T\} \times \bar{\mathcal{S}}. && (5.6.3) \end{aligned}$$

By similar arguments as in Section 5, we can show that $\bar{v}_\varepsilon$ is a constrained viscosity solution to (5.6.2)-(5.6.3), and the following comparison principle holds:
Suppose $u \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$ is a usc viscosity subsolution to (5.6.2)-(5.6.3) on $[0,T] \times \bar{\mathcal{S}}$, and $w \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$ is a lsc viscosity supersolution to (5.6.2)-(5.6.3) on $[0,T] \times \mathcal{S}$, such that

$$u(t,z,\theta) \quad \leq \quad \liminf_{\substack{(t',z',\theta') \to (t,z,\theta) \\ (t',z',\theta') \in [0,T) \times \mathcal{S}}} w(t',z',\theta'), \quad \forall (t,z,\theta) \in [0,T] \times D_0.$$

Then,

$$u \quad \leq \quad w \quad \text{on} \quad [0,T] \times \mathcal{S}. \tag{5.6.4}$$

The proof follows the same lines of arguments as in the proof of Theorem 5.5.2 (the function ψ is still a strict viscosity supersolution to (5.6.2)-(5.6.3) on $[0,T] \times \bar{\mathcal{S}}$), and so we omit it.

As a consequence, we obtain a PDE characterization of the value function v.

Proposition 5.6.2 *The value function v is the minimal constrained viscosity solution in $\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$ to* (5.4.4)-(5.4.5)*, satisfying the boundary condition*

$$\lim_{(t',z',\theta') \to (t,z,\theta)} v(t',z',\theta') \;=\; v(t,z,\theta) \;=\; U(0), \quad \forall (t,z,\theta) \in [0,T] \times D_0. \tag{5.6.5}$$

Proof. Let $V \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$ be a viscosity solution in $\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$ to (5.4.4)-(5.4.5), satisfying the boundary condition (5.6.5). Since $\mathcal{H} \geq \bar{\mathcal{H}}_\varepsilon$, it is clear that V_* is a viscosity supersolution to (5.6.2)-(5.6.3). Moreover, since $\lim_{(t',z',\theta') \to (t,z,\theta)} V_*(t',z',\theta') = U(0) = v(t,z,\theta) \geq \bar{v}_\varepsilon^*(t,z,\theta)$ for $(t,z,\theta) \in [0,T] \times D_0$, we deduce from the comparison principle (5.6.4) that $V \geq V_* \geq \bar{v}_\varepsilon^* \geq \bar{v}_\varepsilon$ on $[0,T] \times \mathcal{S}$. By sending ε to 0, and from the convergence result in Proposition 5.6.1, we obtain: $V \geq v$, which proves the required result. □

Appendix: constrained viscosity solutions to parabolic QVIs

We consider a parabolic quasi-variational inequality in the form:

$$\min\Big[-\frac{\partial v}{\partial t} + F(t,x,v,D_x v, D_x^2 v) \ , \ v - \mathcal{H}v \Big] \;=\; 0, \quad \text{in } [0,T) \times \bar{\mathcal{O}}, \tag{A.1}$$

together with a terminal condition

$$\min\big[v - g \ , \ v - \mathcal{H}v \big] \;=\; 0, \quad \text{in } \{T\} \times \bar{\mathcal{O}}. \tag{A.2}$$

Here, $\mathcal{O} \subset \mathbb{R}^d$ is an open domain, F is a continuous function on $[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d$ ($\mathcal{S}^d$ is the set of positive semidefinite symmetric matrices in $\mathbb{R}^{d \times d}$), nonincreasing in its last argument, g is a continuous function on $\bar{\mathcal{O}}$, and $\mathcal{H}$ is a nonlocal operator defined on the set of locally bounded functions on $[0,T] \times \bar{\mathcal{O}}$ by:

$$\mathcal{H}v(t,x) \;=\; \sup_{e \in \mathcal{C}(t,x)} [v(t,\Gamma(t,x,e)) + c(t,x,e)].$$

$\mathcal{C}(t,x)$ is a compact set of a metric space E, eventually empty for some values of (t,x), in which case we set $\mathcal{H}v(t,x) = -\emptyset$, and is continuous for the Hausdorff metric, i.e. if (t_n,x_n) converges to (t,x) in $[0,T] \times \bar{\mathcal{O}}$, and (e_n) is a sequence in $\mathcal{C}(t_n,x_n)$ converging to e, then $e \in \mathcal{C}(t,x)$. The functions Γ and c are continuous, and such that $\Gamma(t,x,e) \in \bar{\mathcal{O}}$ for all $e \in \mathcal{C}(t,x,e)$.

Given a locally bounded function u on $[0,T] \times \bar{\mathcal{O}}$, we define its lower-semicontinuous (lsc in short) envelope u_* and upper-semicontinuous (usc) envelope u^* on $[0,T] \times \bar{\mathcal{S}}$ by:

$$u_*(t,x) \;=\; \liminf_{\substack{(t',x') \to (t,x) \\ (t',x') \in [0,T) \times \mathcal{O}}} u(t',x'), \qquad u^*(t,x) \;=\; \limsup_{\substack{(t',x') \to (t,x) \\ (t',x') \in [0,T) \times \mathcal{O}}} u(t',x').$$

One can check (see e.g. Lemma 5.1 in [53]) that the operator $\mathcal{H}$ preserves lower and upper-semicontinuity:

(i) $\mathcal{H}u_*$ is lsc, and $\mathcal{H}u_* \leq (\mathcal{H}u)_*$, (ii) $\mathcal{H}u^*$ is usc, and $(\mathcal{H}u)^* \leq \mathcal{H}u^*$.

We now give the definition of constrained viscosity solutions to (A.1)-(A.2). This notion, which extends the definition of viscosity solutions of Crandall, Ishii and Lions (see [23]), was introduced in [76] for first-order equations for taking into account boundary conditions arising in state constraints, and used in [79] for stochastic control problems in optimal investment.

Definition A.1 *A locally bounded function v on $[0,T] \times \bar{\mathcal{O}}$ is a constrained viscosity solution to* (A.1)-(A.2) *if the two following properties hold:*
(i) Viscosity supersolution property on $[0,T] \times \mathcal{O}$: for all $(\bar{t},\bar{x}) \in [0,T] \times \mathcal{O}$, and $\varphi \in C^{1,2}([0,T] \times \mathcal{O})$ with $0 = (v_* - \varphi)(\bar{t},\bar{x}) = \min(v_* - \varphi)$, we have

$$\begin{aligned}
\min\Big[-\frac{\partial \varphi}{\partial t}(\bar{t},\bar{x}) + F(\bar{t},\bar{x},\varphi_*(\bar{t},\bar{x}), D_x\varphi(\bar{t},\bar{x}), D_x^2\varphi(\bar{t},\bar{x}))\ , \qquad & \\
v_*(\bar{t},\bar{x}) - \mathcal{H}v_*(\bar{t},\bar{x})\Big] \;&\geq\; 0, \;\; (\bar{t},\bar{x}) \in [0,T) \times \mathcal{O}, \\
\min\big[v_*(\bar{t},\bar{x}) - g(\bar{x})\ ,\ v_*(\bar{t},\bar{x}) - \mathcal{H}v_*(\bar{t},\bar{x})\big] \;&\geq\; 0, \;\; (\bar{t},\bar{x}) \in \{T\} \times \mathcal{O}.
\end{aligned}$$

(ii) Viscosity subsolution property on $[0,T] \times \bar{\mathcal{O}}$: for all $(\bar{t},\bar{x}) \in [0,T] \times \bar{\mathcal{O}}$, and $\varphi \in C^{1,2}([0,T] \times \bar{\mathcal{O}})$ with $0 = (v^* - \varphi)(\bar{t},\bar{x}) = \max(v^* - \varphi)$, we have

$$\begin{aligned}
\min\Big[-\frac{\partial \varphi}{\partial t}(\bar{t},\bar{x}) + F(\bar{t},\bar{x},\varphi_*(\bar{t},\bar{x}), D_x\varphi(\bar{t},\bar{x}), D_x^2\varphi(\bar{t},\bar{x}))\ , \qquad & \\
v^*(\bar{t},\bar{x}) - \mathcal{H}v^*(\bar{t},\bar{x})\Big] \;&\leq\; 0, \;\; (\bar{t},\bar{x}) \in [0,T) \times \bar{\mathcal{O}}, \\
\min\big[v^*(\bar{t},\bar{x}) - g(\bar{x})\ ,\ v^*(\bar{t},\bar{x}) - \mathcal{H}v^*(\bar{t},\bar{x})\big] \;&\leq\; 0, \;\; (\bar{t},\bar{x}) \in \{T\} \times \bar{\mathcal{O}}.
\end{aligned}$$

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Résumé : Nous étudions le lien entre EDS rétrogrades et certains problèmes d'optimisation stochastique ainsi que leurs applications en finance. Dans la première partie, nous nous intéressons à la représentation par EDSR de problème d'optimisation stochastique séquentielle : le contrôle impulsionnel et le switching optimal. Nous introduisons la notion d'EDSR contrainte à sauts et montrons qu'elle donne une représentation des solutions de problème de contrôle impulsionnel markovien. Nous lions ensuite cette classe d'EDSR aux EDSRs à réflexions obliques et aux processus valeurs de problèmes de switching optimal. Dans la seconde partie nous étudions la discrétisation des EDSRs intervenant plus haut. Nous introduisons une discrétisation des EDSRs contraintes à sauts utilisant l'approximation par EDSRs pénalisées pour laquelle nous obtenons la convergence. Nous étudions ensuite la discrétisation des EDSRs à réflexions obliques. Nous obtenons pour le schéma proposé une vitesse de convergence vers la solution continument réfléchie. Enfin dans la troisième partie, nous étudions un problème de liquidation optimale de portefeuille avec risque et coût d'exécution. Nous considérons un marché financier sur lequel un agent doit liquider une position en un actif risqué. L'intervention de cet agent influe sur le prix de marché de cet actif et conduit à un coût d'exécution modélisant le risque de liquidité. Nous caractérisons la fonction valeur de notre problème comme solution minimale d'une inéquation quasi-variationnelle au sens de la viscosité contrainte.

Mots-clés : EDS rétrogrades contraintes, réflexions obliques, contrôle impulsionnel, switching optimal, solution de viscosité, inégalités variationnelles, discrétisation d'EDSR, risque de liquidité, maximisation d'utilité.

Discipline : Mathématiques

Abstract : We study the link between Backward SDEs and some stochastic optimal control problems and their application to mathematical finance. In the first part we focus on the BSDE representation of solution to impulse control and optimal switching. We first introduce the notion of constrained BSDEs with jumps and prove that it gives a representation of solutions to Markovian impulse control problems. We then bind these contrained BSDEs to BSDEs with oblique reflexion and optimal switching problems. In the secoond part, we study the time discretization of the previous BSDEs. We first state a discretization of constrained BSDE using the approximation given by the penalized BSDEs. We the provide a speed convergence for the natural scheme associated to BSDEs with oblique reflections. Finally, in the third part, we consider a liquidation problem under execution risk and cost. We characterize the associated value function as the minimal solution to the associated quasi-variational inequality.

Key words : Constrained Backward SDEs, impulse control, optimal switching, viscosity solution, variational inégalites, BSDE discrétization, liquidity risk, utility maximization.

Laboratoire de Probabilités et Modèles Aléatoires,
CNRS-UMR 7599, UFR de Mathématiques, case 7012
Université Paris 7, Paris Diderot
2, place Jussieu, 75251 Paris Cedex 05.